SCALE-INVARIANT GRAVITY: GEOMETRODYNAMICS

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Abstract

We present a scale-invariant theory, *conformal gravity*, which closely resembles the geometrodynamical formulation of general relativity (GR). While previous attempts to create scale-invariant theories of gravity have been based on Weyl's idea of a compensating field, our direct approach dispenses with this and is built by extension of the method of best matching w.r.t scaling developed in the parallel particle dynamics paper by one of the authors. In spatially-compact GR, there is an infinity of degrees of freedom that describe the shape of 3-space which interact with a single volume degree of freedom. In conformal gravity, the shape degrees of freedom remain, but the volume is no longer a dynamical variable. Further theories and formulations related to GR and conformal gravity are presented.

Conformal gravity is successfully coupled to scalars and the gauge fields of nature. It should describe the solar system observations as well as GR does, but its cosmology and quantization will be completely different.

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1 Inroduction

During the years that he created special and general relativity, Einstein had several goals [1]. The first, realized in special relativity, was to reconcile Maxwell's wave theory of light with universal validity of the restricted relativity principle (RRP). In contrast to Lorentz, who explicitly sought a constructive theory [2] to explain the Michelson–Morley experiment and the RRP, Einstein was convinced that the quantum effects discovered by Planck invalidated such an approach [1], p. 45. He "despaired of the possibility of discovering the true laws by means of constructive efforts" [1], p. 53, and instead adopted the RRP as an axiomatic principle. Einstein's further goals were the implementation of Mach's principle and the construction of a field theory of gravitation analogous to Maxwellian electromagnetism. Encouraged by his treatment of the RRP as a principle to be adopted rather than a result to be derived, Einstein generalized it to the general relativity principle (GRP), according to which the laws of nature must take an identical form in all frames of reference. The GRP was eventually implemented as the four-dimensional general covariance of a pseudo-Riemannian dynamical spacetime.

In making spacetime the arena of dynamics, Einstein broke radically with the historical development of dynamics, in which the configuration space and phase space had come to play ever more dominant roles. In fact, both of these played decisive roles in the discovery of quantum mechanics, especially the symplectic invariance of Hamiltonian dynamics on phase space. Since then, Hamiltonian dynamics has also played a vital role in the emergence of modern gauge theory [4]. In fact, spacetime and the canonical dynamical approach have now coexisted for almost a century, often creatively but also not without tension.

This tension became especially acute when Dirac and Arnowitt, Deser and Misner (ADM) [4, 5] reformulated the Einstein field equations as a constrained Hamiltonian dynamical system describing the evolution of Riemannian 3-metrics g_{ij}:

$$\mathsf{H} = \int \mathrm{d}^3x (N\mathcal{H} + \xi^i \mathcal{H}_i) \tag{1}$$

$$\mathcal{H} \equiv G_{ijkl} p^{ij} p^{kl} - \sqrt{g} R = 0, \tag{2}$$

$$\mathcal{H}_i \equiv -2 \nabla_j {p_i}^j = 0, \tag{3}$$

where a divergence term has been omitted from (1). The 3-metric g_{ij}[1] has determinant g and conjugate momentum p^{ij}, N is the lapse, ξ^i is the shift, R is the 3-dimensional Ricci scalar, ∇_j is the 3-dimensional covariant derivative, $G_{ijkl} = \frac{1}{\sqrt{g}}(g_{i(k}g_{l)j} - \frac{1}{2}g_{ij}g_{kl})$ is the DeWitt supermetric [6], and $\mathcal{H}$, $\mathcal{H}_i$ are the algebraic scalar Hamiltonian constraint and differential vector momentum constraint respectively. Dirac was so impressed by the simplicity of the Hamiltonian formulation that he questioned the status of spacetime [7], remarking 'I am inclined to believe ... that four-dimensional symmetry is not a fundamental property of the physical world.' Wheeler too was struck by this development and coined the words *geometrodynamics* for the Einsteinian evolution of 3-dimensional Riemannian geometries (*3-geometries*) embedded in spacetime and *superspace*, the configuration space formed by all 3-geometries on a given 3-manifold $\mathcal{M}$ [8].

Mathematically, superspace is obtained from Riem, the space of all (suitably continuous) Riemannian 3-metrics g_{ij} defined on $\mathcal{M}$ (taken here – as an important physical assumption – to be compact and without boundary) by quotienting with respect to 3-dimensional diffeomorphisms on

[1] In this paper, we use lower case Latin letters for spatial indices and upper case Latin letters for internal indices. We use round brackets for symmetrization and square brackets for antisymmetrization; indices to be excluded from (anti)symmetrization are set between vertical lines.

$\mathcal{M}$:

$$\{\text{Superspace}\} = \frac{\{\text{Riem}\}}{\{\text{3-Diffeomorphisms}\}}. \tag{4}$$

Superspace is the analogue of the relative configuration space discussed in the particle-dynamics paper [9]. [2]

The ADM configuration space is the extension of Riem to Riem $\times\, \Xi \times P$, where Ξ is the space of the ξ^i and $N \in P$, the space of (suitably differentiable) positive functions. However, since N and ξ^i have no conjugate momenta, the true gravitational degrees of freedom of GR are contained in Riem. These degrees of freedom are furthermore subjected to the Hamiltonian and momentum constraints (2) and (3). If, in the thin-sandwich problem (Sec. PD.4) for geometrodynamics (Sec. 2), one could solve the momentum constraint in terms of the Lagrangian variables for ξ^i, then the theory, now formally defined on Riem, would actually be defined on superspace because the physical degrees of freedom have been reduced to three. However, on account of the still remaining Hamiltonian constraint (2), superspace must still contain one redundant degree of freedom per space point.

The hitherto unresolved status of this one remaining redundancy, which can only be eliminated at the price of breaking the spacetime covariance of GR, is probably the reason why neither Dirac nor Wheeler, despite being struck by the Hamiltonian structure of GR, made any serious subsequent attempt to do without the notion of spacetime. In particular, Wheeler formulated the idea of embeddability, i.e, that Riemmanian 3-geometries always evolve in such a way that they can be embedded in a four-dimensional pseudo-Riemannian spacetime [8]. This idea led Hojman, Kuchař and Teitelboim [10] to a new derivation of general relativity as a constrained Hamiltonian system. However, this was not a purely dynamical derivation, since the embeddability condition played a crucial role.

In the recent [11], called henceforth RWR from its title 'Relativity Without Relativity', and [12], we showed that GR, the universal light cone of special relativity, and the gauge principle could all be derived in a unified manner using principles that in no way presuppose spacetime. Our so-called 3-space approach developed out of an earlier attempt [13] to implement Mach's idea of a relational dynamics directly in a constructive dynamical theory, in contrast to Einstein's indirect approach through generalization of the RRP to the GRP. For the direct approach, superspace is the natural relational arena for geometrodynamics. A detailed motivation is given in [11] and PD. The aim of this paper is to extend the techniques of [11] to include scaling invariance and thereby push the idea of relational dynamics to its logical conclusion. This has already been achieved in PD for the Newtonian context of point particles that interact in Euclidean space. The present paper extends the scaling techniques developed in PD to geometrodynamics. This adds to the aforementioned tension, since it provides an example of a theory of evolving 3-geometries that is not generally covariant but couples satisfactorily to the accepted bosonic fields of nature and should pass the solar-system and binary-pulsar tests of GR.

As explained further in PD, the aim of the 3-space approach is, first, to identify the configuration space Q_0 of the true physical degrees of freedom and, second, to formulate the dynamics in such a way that specification of an initial point q_0 in Q_0 and an initial *direction* at q_0 suffices to determine a unique dynamical curve in Q_0. This relational geodesic principle is implemented by *best matching*, which is explained in detail in [11] and PD, using an action that is *homogeneous of degree one in the velocities*.

Both properties of the action – the best matching and the homogeneity – lead to constraints. An important tool of the 3-space approach is Dirac's generalized Hamiltonian dynamics [4], used to

[2]This paper is henceforth referred to as PD. References to it are identified by PD followed by the relevant section or equation number.

ensure that all the constraints are propagated by the Euler–Lagrange equations. The precise form of the best matching, which always leads to constraints linear and homogeneous in the canonical momenta, is either clear in advance, being determined by the symmetry used in the quotienting to obtain the physical configuration space Q_0 [a good example is the quotienting by 3-diffeomorphisms, which leads to superspace (4) and the momentum constraint (3)] or is dictated by the emergence of a constraint when the Dirac consistency procedure is used (this is how gauge theory arises of necessity and is then encoded in the action in RWR [11, 12]).

For the homogeneity, there is an important freedom in the manner in which it is implemented. In PD, the Lagrangian is a simple square root of an expression quadratic in the velocities. In [11, 12] and the present paper, a *local square root* is used. This means that the Lagrangian is an integral over the manifold $\mathcal{M}$ of the square root of a quadratic expression calculated at each point of $\mathcal{M}$ before the integration is performed. The local square root leads to infinitely many quadratic constraints, one per point of $\mathcal{M}$. This is in marked contrast to the single 'global' quadratic constraint that follows from Jacobi-type actions in particle mechanics.

In RWR [11], the local square root is in the Baierlein–Sharp–Wheeler (BSW) form [14] for the action for GR,

$$S_{\mathrm{BSW}} = \int \mathrm{d}\lambda \int \mathrm{d}^3x \sqrt{g}\sqrt{R}\sqrt{T}, \tag{5}$$

where the kinetic term

$$T = \frac{1}{\sqrt{g}} G^{abcd} \frac{\mathrm{d}g_{ab}}{\mathrm{d}\lambda}\frac{\mathrm{d}g_{cd}}{\mathrm{d}\lambda} \tag{6}$$

is constructed by the best-matching correction of the velocities to allow for the action of 3-diffeomorphisms (which generalize simultaneously the translations and rotations considered for particles in Euclidean space in PD)[3]

$$\frac{\mathrm{d}g_{ab}}{\mathrm{d}\lambda} \equiv \frac{\partial g_{ab}}{\partial\lambda} - (\xi^c\partial_c g_{ab} + g_{ac}\partial_b\xi^c + g_{bc}\partial_a\xi^c) = \frac{\partial g_{ab}}{\partial\lambda} - \pounds_\xi g_{ab}, \tag{7}$$

and the inverse DeWitt supermetric $G^{abcd} = \sqrt{g}(g^{ac}g^{bd} - g^{ab}g^{cd})$. Here, $\pounds_\xi$ is the Lie derivative w.r.t ξ_i. We often abbreviate $\frac{\partial}{\partial\lambda}$ by a dot.

The 3-space approach works in RWR [11] as follows. One starts with an action of the same form as (5) but with the 3-scalar curvature R replaced by an arbitrary scalar concomitant of the 3-metric g_{ij} and the general inverse supermetric $G^{ijkl}_{\mathrm{W}} = \sqrt{g}(g^{ik}g^{jl} - Wg^{ij}g^{kl})$ for W constant ($W \neq \frac{1}{3}$ is required for invertibility). At each point of $\mathcal{M}$, a scalar algebraic constraint and a vector differential constraint must be satisfied. The scalar constraint arises purely from the local square-root form of any BSW-type Lagrangian, is quadratic in the canonical momenta and has the general form of the quadratic ADM Hamiltonian constraint $\mathcal{H}$ (2). The vector constraint arises from variation w.r.t the auxiliary variable ξ^i and is identical to the ADM momentum constraint $\mathcal{H}_i$ (3). This universal form arises because best matching entails replacement of the 'bare' velocity $\partial g_{ij}/\partial\lambda$ by the corrected velocity $\mathrm{d}g_{ij}/\mathrm{d}\lambda$ (7), the form of which is uniquely determined by the diffeomorphism symmetry. One then checks whether the modified Euler–Lagrange equations propagate the constraints. In general, they do not, and to arrive at a consistent theory one must either introduce new constraints as outlined by Dirac [4] or drastically limit the modifications of the BSW action. However, new constraints will rapidly exhaust the degrees of freedom, which have already been reduced to two per space point by the existing constraints. This may be called Dirac's method by exhaustion,

[3] As in PD, we shall use the formalism of corrected coordinates and corrected velocities to treat conformal transformations. For the 3-diffeomorphisms, we shall use only corrected velocities, denoting them, as in (7), by the 'straight' d. Note also that in this paper we use the widely employed ADM sign conventions and notation, which results in minor differences compared with the expressions in RWR [11].

and it is very powerful. In RWR [11] it was shown to enforce $W = 1$ and to limit the possible modifications of R to $sR + \Lambda$, where Λ is a cosmological constant and $s = 0, 1, -1$, corresponding to so-called strong gravity [15] (for which $W \neq 1$ is allowed [16]), and Lorentzian and Euclidean GR, respectively. This is the 'relativity without relativity' result. This and other modifications of the action were considered in [11, 17, 16]. Consistency also exhaustively forces classical bosonic theories to have the form of the currently known bosonic gauge fields and to respect a universal light cone [11, 12, 18].

In this paper, we construct a new scale-invariant theory of gravity by employing best matching not only w.r.t 3-diffeomorphisms but also w.r.t 3-dimensional conformal transformations:

$$g_{ij} \longrightarrow \omega^4 g_{ij}, \tag{8}$$

where the scalar ω is an arbitrary smooth positive function of the label λ and of the position on $\mathcal{M}$. The transformation (8) is the infinite dimensional 'localization' of the first member of the scaling transformation (Eq. PD.23). (The fourth power of ω is traditional and is used for computational convenience.) The new theory, which we call *conformal gravity*, is a consistent best-matching generalization of the BSW action invariant under (8). It was proposed in the brief communication [17], which, however, treated only pure gravity and was written before the development of the special variational technique with free end points described in PD. The present paper uses the new method and develops three aspects: the Hamiltonian formulation, the coupling to the known bosonic matter fields and the physical and mathematical implications of conformal gravity.

In Hamiltonian GR, the six degrees of freedom per manifold point present in Riem are reduced to two by the differential vector constraint (3) and the algebraic scalar Hamiltonian constraint (2). These constraints arise from variation w.r.t to the lapse N and shift ξ^i. The lapse and the shift remain completely free gauge variables and can be chosen at will in the course of evolution. They are velocities in unphysical gauge directions.

In conformal gravity, we retain the basic form of the BSW action with local square root but replace g_{ij} by the corrected coordinates

$$\bar{g}_{ij} = \phi^4 g_{ij} \tag{9}$$

and add the best matching w.r.t the conformal factor ϕ in (9). This leads to a second algebraic scalar constraint that holds at each space point:

$$\mathrm{tr}p^{ij} \equiv p^{ij} g_{ij} \equiv p = 0, \tag{10}$$

which, unlike the quadratic Hamiltonian constraint, is *linear* in the canonical momenta. The trace p in conformal gravity is analogous to the dilatational momentum defined by (Eq. PD.3). However, since (8) is a local transformation, in contrast to the global transformation (Eq. PD.23), we have one conformal constraint at each space point as with the Hamiltonian and momentum constraints. Just as the vanishing dilatational momentum in particle dynamics conserves the moment of inertia I of an island universe, here the vanishing trace conserves the volume of a spatially compact universe.

Since conformal gravity augments the ADM Hamiltonian (2) and momentum (3) constraints by the conformal constraint (10), a simple count suggests that the new theory should have only a single true degree of freedom per space point, in contrast to the two present in Hamiltonian GR. However, this is not the case. We show in Sec. 3 that the free-end-point variation w.r.t the conformal factor ϕ leads not only to the constraint (10) but also to a further condition that fixes the analogue in conformal gravity of the lapse N in GR. Whereas in GR N and ξ^i are freely specifiable gauge velocities, in conformal gravity N is fixed, and its role as gauge variable is taken over by ϕ. Since ξ^i plays the same role in both theories, conformal gravity, like GR, has two degrees of freedom per space point. However, in contrast to GR, they are unambiguously identified as the two conformal shape degrees of freedom of the 3-metric g_{ij}. Conceptually, this is a pleasing result, but it has a far-reaching consequence – conformal gravity cannot be cast into the form of a four-dimensionally generally covariant spacetime theory. Because the lapse is fixed, absolute simultaneity and a preferred frame of reference are introduced.

The reader may feel that this is too high a price to pay for a scale-invariant theory. Of course, experiment will have the final word. However, one of the aims of this paper is to show that conformal invariance already has the potential to undermine the spacetime covariance of GR. In order to demonstrate this, in Sec. 2 we explain York's powerful conformal method [19, 20, 21, 22] for finding initial data that satisfy the constraints (2) and (3). This serves three purposes: it introduces invaluable concepts and techniques, it establishes an intimate connection between GR and conformal gravity, and it facilitates the testing of conformal gravity as a putative description of nature, since the two theories are most easily compared using York's techniques.

York's work was stimulated in part by Wheeler's desire to find the true degrees of freedom of GR and with them the physical configuration space of geometrodynamics. For superspace still contains redundancy since it possesses three degrees of freedom per space point whereas GR has only two. In answer to Wheeler's question "What is two thirds of superspace?", York responded that there is only one simple and natural answer. He noted that one can decompose an arbitrary 3-metric g_{ij} into its determinant g and its scale-free part

$$\hat{g}_{ij} \equiv g^{-\frac{1}{3}} g_{ij}, \tag{11}$$

which is invariant under the conformal transformation (8) [21]. York argued that g should be regarded as an unphysical gauge degree of freedom, the elimination of which would remove the final redundancy from GR. Though he continued to work in spacetime, this led him to parametrize the initial data for GR in *conformal superspace* (CS) [22], which is obtained by quotienting Riem by both 3-diffeomorphisms and conformal transformations (8)

$$\mathrm{CS} \equiv \big\{\text{Conformal Superspace}\big\} = \frac{\big\{\text{Riem}\big\}}{\big\{\text{3-Diffeomorphisms}\big\}\big\{\text{Conformal Transformations}\big\}}. \tag{12}$$

Conformal superspace has been studied more recently by Fischer and Moncrief [23].

As summarized in Sec. 2, York was able to represent GR (strictly a large subset of its solutions for the spatially compact case) as a dynamical scheme in which the infinitely many local shape degrees of freedom represented by $\hat{g}_{ij}$ interact with each other and with one single global variable, which is the total volume V of 3-space:

$$V = \int g^{\frac{1}{2}} \mathrm{d}^3 x. \tag{13}$$

Because of this extra variable, we argue that spatially compact GR does not have variables in CS but rather in the marginally larger space obtained by adjoining the volume V to CS, which we call *Conformal Superspace + Volume* and abbreviate by CS+V. It is formally obtained from Riem by quotienting by 3-diffeomorphisms and by *volume-preserving* conformal transformations:

$$\mathrm{CS{+}V} \equiv \big\{\text{Conformal Superspace + Volume}\big\} = \frac{\big\{\text{Riem}\big\}}{\big\{\text{3-Diffeomorphisms}\big\}\big\{\text{Volume-Preserving Conformal Transformations}\big\}}. \tag{14}$$

The introduction of CS enables us to formulate conformal gravity in Sec. 3 as *a best-matching geodesic theory on conformal superspace.* It determines unique curves in CS given an initial point and an initial direction in CS.

In Sec. 4, using the Hamiltonian formulation, we show that spatially compact GR in York's representation corresponds to a closely analogous best-matching theory in CS+V: given an initial point and an initial direction in CS+V, a unique dynamical curve is determined. In both theories, there is a unique definition of simultaneity. The only difference between them is that in conformal gravity the 3-volume (13) is no longer a dynamical variable but a conserved quantity. Although the equations of the best-matching interpretation of GR in CS+V are identical to York's equations, there is a difference. The important CMC condition (Sec. 2) no longer corresponds to a gauge fixing, as hitherto assumed, but to a physical condition as fundamental as the ADM momentum constraint. This will have implications for canonical quantum gravity. The reinterpretation of a subset of GR solutions as solutions of a best-matching theory in CS+V may have more interest in the long term than the fully scale-invariant conformal gravity. This is because the latter is much more vulnerable to experimental disproof than GR; the CS+V theory may have a longer 'shelf life'. Further theories which share some features with GR and conformal gravity are also presented, including the asymptotically flat counterparts of conformal gravity and the CS+V theory.

There are two important differences between the manner in which conformal covariance is achieved in conformal gravity and the two best known earlier attempts to create conformally covariant theories: Weyl's 1917 theory [24] discussed in PD and Dirac's simplified modification of it [25]. First, both of these earlier theories are spacetime theories, and their conformal covariance leaves four-dimensional general covariance intact. In conformal gravity and York's representation of GR, this is not so. Second, the conformal covariance is achieved in the theories of Weyl and Dirac through a compensating field that is conformally transformed with the gravitational field. In Weyl's theory, the compensating field is a 4-vector field that Weyl identified as the electromagnetic field until Einstein [26] pointed out the difficulty discussed in PD Sec. 1. Weyl later reinterpreted the idea of a compensating field in his effective creation of gauge theory [27], but he never salvaged his original theory. In Dirac's simplification, the compensating field is the additional scalar field in Brans–Dicke theory [28]. This possibility has been exploited more recently in theories with a dilatonic field [29]. In contrast, conformal gravity has no physical compensating field; the variable ϕ (9) is a purely auxiliary gauge variable used to implement conformal best matching. This is therefore a more radical approach, in which full scale and conformal invariance of the gravitational field by itself is achieved.[4]

We should also mention that the Lagrangian of conformal gravity possesses *two* homogeneity properties. It is homogeneous of degree one in the velocities and of degree zero in the auxiliary variable ϕ. Just as scaling fixes the potential in PD to be homogeneous of degree -2, it plays an important role in fixing the form of conformal gravity, especially when we include a cosmological constant (Sec. 5) and classical bosonic matter (in Sec. 7 via general theorems proved in Sec. 6). It turns out, as in RWR, that a universal light cone, electromagnetism and Yang–Mills theory are enforced. In Sec. 8, we conclude that conformal gravity should be in agreement with the standard tests of GR. Strong differences will emerge in cosmology and the quantum theory; these will be less pronounced for the CS+V theory.

[4] As we shall show, one of the features of conformal gravity that ensures this is the fact that the Lagrangian depends on *ratios* of the components of the 3-metric. In checking the literature on Weyl's theory, we came across Einstein's 1921 paper [30]. In it, he follows Weyl in employing only ratios of the 4-metric components, but drops the idea of a compensating field. His aim is therefore similar to ours – the gravitational field should be made scale invariant by itself. Einstein's proposal was very tentative, and it seems to us that it must lead to a theory with no nontrivial solutions. We are not aware that he or anyone else attempted to take the idea further.

2 Summary of York's Work

It is well known that the Hamiltonian and momentum constraints (2) and (3) are, respectively, the 00 and $0i$ members of the Einstein field equations (Efe's) and are initial-value constraints. The question arises of how one can obtain a 3-metric and associated canonical momenta that satisfy the constraints. A first method, which corresponds to the logic of the best-matching procedure and uses Lagrangian variables, was proposed by BSW in the same paper [14] in which they presented the BSW action (5). Freely specify g_{ij}, $\dot{g}_{ij}$ and try to solve for ξ^i. If this can be done, the problem is solved, since the 00 Efe in this approach is not an equation but a square-root identity. Trying to solve the $0i$ Efe's for ξ^i is the thin-sandwich conjecture [31]. Although progress has been made, a regular method to solve this has not been found, and counterexamples exist.

A second method is York's. He found solutions to the constraints by working with the Hamiltonian variables g_{ij}, p^{ij}. Instead of using the definition of the canonical momenta

$$p^{ij} = \frac{\partial L}{\partial(\partial g_{ij}/\partial\lambda)} = \frac{\sqrt{g}}{2N} g^{ai} g^{bj} \frac{\mathrm{d}g_{ab}}{\mathrm{d}\lambda} \tag{15}$$

to express the constraints explicitly in terms of Lagrangian variables and the unknown ξ^i, he simply tried to find functions g_{ij} and p^{ij} that satisfy (2) and (3). In this approach, (2) is a proper equation and not an identity. Indeed, what happens in York's method is that one starts by specifying arbitrarily a pair of 3×3 symmetric tensors $\tilde{g}_{ij}$ and $\tilde{p}^{ij}$. Then there exists a regular method to construct from $\tilde{p}^{ij}$ a p^{ij} that satisfies the momentum constraint (3) with respect to $\tilde{g}_{ij}$. Once this has been achieved, one performs a conformal transformation of $\tilde{g}_{ij}$ into g_{ij} in such a way that the new 3-metric satisfies (2). The end result is that the transformed pair g_{ij}, p^{ij} satisfies both constraints. The details are as follows.

In GR, an embedded hypersurface is *maximal* if the trace

$$p^{ij} g_{ij} \equiv p = 0 \tag{16}$$

everywhere on it. Under these circumstances [20], the momentum constraint (3) is invariant under the conformal transformation

$$g_{ij} \longrightarrow \tilde{g}_{ij} \equiv \phi^4 g_{ij}, \; p^{ij} \longrightarrow \tilde{p}^{ij} \equiv \phi^{-4} p^{ij}, \tag{17}$$

whilst the Hamiltonian constraint (2) becomes the Lichnerowicz equation [32]

$$8\nabla^2\phi + M\phi^{-7} - R\phi = 0, \; gM = p_{ij}p^{ij} \geq 0. \tag{18}$$

The fourth power of the positive function ϕ (the conformal factor) is chosen to simplify the appearance of (18). York's method is to solve (3) in a conformally-invariant way and then to take (18) to determine ϕ for $\tilde{g}_{ij}$ given.

An embedded hypersurface has *constant mean curvature* (CMC)[5] if

$$\tau = \frac{2p}{3\sqrt{g}} = \text{spatial constant}. \tag{19}$$

York was able to generalize his method from maximal to CMC hypersurfaces [20, 21]. This generalization works because (19) and (3) imply that $\nabla_b(p^{ab} - \frac{p}{3}g^{ab}) = 0$ and so $\sigma^{ab} \equiv p^{ab} - \frac{p}{3}g^{ab}$ is TT

[5] The term CMC in GR refers to a constant trace K of the extrinsic curvature K_{ab}, which is related to the canonical momentum by $K^{ab} = \frac{1}{\sqrt{g}}\left(p^{ab} - \frac{p}{2}g^{ab}\right)$ and is a measure of how much the hypersurface is bent within spacetime; we will not use K_{ab} in this paper since it is a spacetime concept.

(transverse traceless), a property that York shows is invariant under conformal transformations. Then we can find conformally-related variables $\tilde{g}_{ij}$, $\tilde{\sigma}^{ij}$ which obey

$$\frac{1}{\sqrt{\tilde{g}}}\left(\tilde{p}_{ab}\tilde{p}^{ab}-\frac{1}{2}\tilde{p}^2\right)-\sqrt{\tilde{g}}\tilde{R}=0, \tag{20}$$

and the required scale factor solves the extended Lichnerowicz equation [21]

$$8\nabla^2\phi+M\phi^{-7}-R\phi-\frac{3}{8}\tau^2\phi^5=0\ ,\quad gM=\sigma_{ab}\sigma^{ab}\geq 0 \tag{21}$$

It is important in deriving this equation that τ and so $p/\sqrt{g}$ transforms as a conformal scalar: $\tilde{\tau}=\tau$. If $\tilde{g}_{ij}$ is specified arbitrarily and p^{ij} satisfies (3), York was able to show that the evolution of the conformal geometry and the scale factor ϕ is uniquely determined.

Much of the strength of York's technique arises from the fact that given a 3-metric $\tilde{g}_{ij}$ and an arbitrary $\tilde{p}^{ij}$ there exists a well-proven method to determine the part of $\tilde{p}^{ij}$ that is TT with respect to $\tilde{g}_{ij}$. This solution to the first half of the problem is not lost by the subsequent conformal transformation of $\tilde{g}_{ij}$ determined by the solution of a Lichnerowicz equation, which is remarkably well-behaved. York's method is furthermore robust to the inclusion of source fields [33]. We think it significant that the only known robust method of solving the initial-value constraints of generally covariant GR makes no use of spacetime techniques but rather employs 3-space conformal structures.

In GR, one regards (16) and (19) as the maximal and CMC *gauge conditions* respectively. The CMC slicing yields a foliation that is extremely convenient in the case of globally hyperbolic spacetimes with compact 3-space. The foliation is unique [34, 35], and the value of τ increases monotonically, either from $-\infty$ to ∞ in the case of a Big-Bang to Big-Crunch cosmological solution or from $-\infty$ to zero in the case of eternally expanding universes. In the first case, the volume of the universe increases monotonically from zero to a maximum expansion, at which the maximal condition (16) is satisfied, after which it decreases monotonically to zero. In spatially compact solutions of GR, the total spatial volume cannot remain constant except momentarily at maximum expansion, when $\tau\propto p=0$. Thus, in GR the volume is a dynamical variable. Other authors have noticed that these properties of τ allow its interpretation as a notion of time, the extrinsic *York time* [36].

It is important to distinguish between a single initial use of the CMC slicing condition in order to find consistent initial data and subsequent use of the slicing when the obtained initial data are propagated forward. This is by no means obligatory. The Efe's are such that once consistent initial data have been found they can be propagated with freely specified lapse and shift. This is precisely the content of four-dimensional general covariance. If one should wish to maintain the CMC slicing condition during the evolution, it is necessary to choose the lapse N in such a way that it satisfies the CMC slicing equation

$$2\left(\frac{N}{g}p_{ij}p^{ij}-\nabla^2N\right)-\frac{Np^2}{2g}=C=\frac{\partial}{\partial\lambda}\left(\frac{p}{\sqrt{g}}\right), \tag{22}$$

where C is a spatial constant. In this paper, we refer to such equations as *lapse-fixing equations*, because, since we do not presuppose GR, we do not always work in a context where the notion of slicing makes sense. Being homogeneous, (22) does not fix N uniquely but only up to global reparametrization

$$N\longrightarrow f(\lambda)N, \tag{23}$$

where $f(\lambda)$ is an arbitrary monotonic function of λ. Similarly, to maintain the maximal gauge condition (16), which can only be done in the spatially non-compact case, N must satisfy the maximal lapse-fixing equation

$$\frac{N}{g} p^{ab} p_{ab} - \nabla^2 N = 0. \tag{24}$$

Now, as we show in Sec. 3, best matching w.r.t the conformal transformations (8) enforces the maximal condition $p = 0$. We also show there that this means that in conformal gravity the total spatial volume V of the universe remains constant. Moreover, its solutions will strongly resemble solutions of GR in the CMC foliation at maximum expansion. The parallel with GR is actually even closer than this. As was emphasized in PD, best matching requires the variation w.r.t the auxiliary best-matching variable to be performed subject to free end points. As a result, it leads to two conditions, not one. In the case of conformal gravity, the conformal best matching w.r.t ϕ leads to the constraint $p = 0$ and to a further condition that ensures propagation of $p = 0$. The latter is identical to the CMC slicing equation in GR except for the modification needed to maintain $p = 0$ rather than the marginally weaker (19).

In Sec. 4, we consider whether GR should be regarded as a four-dimensionally generally covariant spacetime theory or as a 3-space theory that determines unique dynamical curves in CS+V. Here, we merely remark that the CMC condition has hitherto been regarded as a gauge fixing without fundamental physical significance. We believe that our derivation in Sec. 3 of very close parallels of both CMC conditions by conformal best matching casts new light on this issue and supports York's contention that the two conformal shape degrees of freedom in the 3-metric really are, together with the volume V, the true dynamical degrees of freedom in spatially compact GR.

We have already noted that there are then interpretational differences. First, not all GR spacetimes are CMC sliceable, nor is a CMC slicing necessarily extendible to cover the maximal analytic extension of a spacetime [37]. Second, effects regarded as gauge artifacts in GR, such as the 'collapse of the lapse' $N \longrightarrow 0$ in gravitational collapse [37, 35] will be physical effects in theories in which CMC slicing has its origin in a fundamental principle.

3 Lagrangian Formulation of Conformal Gravity

The basic idea of best-matching actions has been fully explained in RWR and PD, so here we merely need to present its application to conformal transformations. We work in the large space Riem, which contains redundancy. To each 3-metric g_{ij} on a compact 3-manifold we can apply diffeomorphisms and conformal transformations. They generate the infinite-dimensional orbit of g_{ij}, which has four dimensions per point of the 3-manifold, three corresponding to the diffeomorphisms and one to the conformal transformations. In the introduction, we have already explained how the conformal constraint takes over the role of the Hamiltonian constraint in GR in restricting by one the degrees of freedom. The number of physical degrees of freedom will therefore be two, as it is in GR. Let us now see how this works out in practice.

We are looking for a consistent action along the lines of the BSW local square-root action $S_{\mathrm{BSW}} = \int \mathrm{d}\lambda \int \mathrm{d}^3x \sqrt{g}\sqrt{R}\sqrt{T}$, which is additionally to be conformally best-matched.

If we introduce the λ-dependent conformal transformation

$$g_{ab} \longrightarrow \omega^4 g_{ab}, \tag{25}$$

the velocity becomes

$$\frac{\mathrm{d}(\omega^4 g_{ab})}{\mathrm{d}\lambda} \equiv \omega^4 \left(\frac{\mathrm{d}g_{ab}}{\mathrm{d}\lambda} + \frac{4}{\omega} g_{ab} \left(\frac{\partial \omega}{\partial \lambda} - \xi^c \partial_c \omega \right) \right) \equiv \omega^4 \left(\frac{\mathrm{d}g_{ab}}{\mathrm{d}\lambda} + \frac{4}{\omega} g_{ab} \frac{\mathrm{d}\omega}{\mathrm{d}\lambda} \right). \tag{26}$$

By analogy with the particle model (Sec. PD.4), we now introduce the corrected coordinates [6]

$$\bar{g}_{ij} = \phi^4 g_{ij}. \tag{27}$$

They are trivially invariant under the pair of compensating transformations

$$g_{ab} \longrightarrow \omega^4 g_{ab}, \quad \phi \longrightarrow \frac{\phi}{\omega}, \tag{28}$$

which is the conformal generalization of the banal transformation (Eq. PD.23).

Under such a transformation, the kinetic scalar

$$T_{\mathrm{W}}^{\mathrm{C}} = \phi^{-8} G_{\mathrm{W}}^{abcd} \frac{\mathrm{d}(\phi^4 g_{ab})}{\mathrm{d}\lambda} \frac{\mathrm{d}(\phi^4 g_{cd})}{\mathrm{d}\lambda}, \tag{29}$$

deduced from the form of (26), is invariant.[7] Note that we begin with an arbitrary W in the inverse supermetric $G_{\mathrm{W}}^{abcd} = \sqrt{g}(g^{ac}g^{bd} - W g^{ab}g^{cd})$. However, whereas $W = 1$ was found in RWR [11] to be crucial for the consistency of GR, we will see that this apparent freedom does not play any role in conformal gravity.

In contrast to (29) and the situation in relativity [11], each term in the potential part of a generalized BSW action changes under (25):

$$\sqrt{g} \longrightarrow \omega^6 \sqrt{g}, \quad R \longrightarrow \omega^{-4}\left(R - \frac{8\nabla^2\omega}{\omega}\right), \tag{30}$$

with the consequence that the action of conformal gravity must, as a 'conformalization' of the BSW action, depend not only on the velocity $\dot{\phi}$ of the auxiliary variable ϕ but also on ϕ itself. This is what we found for the gauge variable a of the dilatation-invariant particle action (Eq. PD.24).

On the basis of (30), the action for pure (matter-free) conformal gravity is

$$S = \int \mathrm{d}\lambda \int \mathrm{d}^3x \left(\left(\frac{\sqrt{g}\phi^6}{V} \right) \sqrt{\left(\frac{V^{\frac{2}{3}}}{\phi^4} \right) \left(R - \frac{8\nabla^2\phi}{\phi} \right)} \sqrt{T_{\mathrm{W}}^{\mathrm{C}}} \right) = \int \mathrm{d}\lambda \frac{\int \mathrm{d}^3x \sqrt{g}\phi^4 \sqrt{R - \frac{8\nabla^2\phi}{\phi}} \sqrt{T_{\mathrm{W}}^{\mathrm{C}}}}{V^{\frac{2}{3}}}, \tag{31}$$

where V is the 'conformalized' volume

$$V = \int \mathrm{d}^3x \sqrt{g}\phi^6. \tag{32}$$

We first explain why the volume V appears in (31).[8] Let the Lagrangian density in (31) be

$$\mathcal{L} = \mathcal{L}\left(g_{ij}, \frac{\mathrm{d}g_{ij}}{\mathrm{d}\lambda}, \phi, \frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right), \qquad S = \int \mathrm{d}\lambda \int \mathrm{d}^3x \mathcal{L}.$$

[6] The introduction of ϕ doubles the conformal redundancy. The 3-diffeomorphism redundancy has already been doubled by ξ^i. The action (31) of conformal gravity will be defined below on Riem $\times\ \Xi \times P$, where $\xi^i \in \Xi$ and the conformal factor $\phi \in P$, the space of positive functions. The original fourfold redundancy per space point of Riem is thereby doubled.

[7] Because we are using a hybrid formalism – correcting both coordinates and velocities under conformal transformations but only the velocities under the diffeomorphisms – we shall not explicitly employ barred variables in the way they are in PD. (The reason why only the velocities need a diffeomorphism correction is that the tensor calculus yields the convenient scalar densities $\sqrt{g}$ and $\sqrt{g}\sqrt{R}$, which are functions of the g_{ij} (and their spatial derviatives) alone. This matches the implementation of (Eq. PD.39) by translationally invariant potentials. There are no simple analogous invariants under (25), as we shall see immediately.) Note also that the rule of transformation of the diffeomorphism auxiliary variable under (28) leaves ξ^i unchanged and changes ξ_i.

[8] A consequence of the appearance of the volume in the action is that whereas the GR BSW action has intrinsic physical dimensions of length squared, the conformal gravity action is dimensionless (see also Sec. PD.6). The significance of this for the emergence of the equivalent notion to Newton's gravitational constant will be considered in subsequent work on the weak field limit of conformal gravity.

Provided we ensure that $\mathcal{L}$ is a functional of the corrected coordinates and velocities, it is bound to be invariant under the conformal transformations (28). This would be the case if we omitted the volume V in (31). But by the rules of best matching formulated in Sec. PD.4 the Lagrangian must also be invariant separately under all possible variations of the auxiliary variable. Now in this case the auxiliary ϕ is a function of position, so one condition that $\mathcal{L}$ must satisfy is

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \text{ if } \phi = \text{spatial constant.} \tag{33}$$

A glance at (31) shows that if V were removed, (33) could not be satisfied. The action must be homogeneous of degree zero in ϕ.[9] We exhibit this in the first expression for $\mathcal{L}$ in (31), in which the two expressions (30) that derive from $\sqrt{g}$ and $\sqrt{R}$ in the BSW action have been multiplied by appropriate powers of V. (The second expression is more convenient for calculations.)

A separate issue is the means of achieving homogeneity. We originally attempted to achieve homogeneity by using not $R^{1/2}$ but $R^{3/2}$, since $\sqrt{g}R^{3/2}$ will satisfy (33). However, this already leads to an inconsistent theory even before one attempts 'conformalization' [11]. Since an ultralocal kinetic term T has no conformal weight, another possibility would be to construct a conformally-invariant action by multiplying such a T by a conformally-invariant 3-dimensional scalar density of weight 1. Whereas the Bach tensor and the Cotton–York tensor are available conformally-invariant objects [19], the suitable combinations they give rise to are cumbersome, and much hard work would be required to investigate whether any such possibilities yield consistent theories. Even if they do, they will certainly be far more complicated than conformal gravity.

We have therefore simply used powers of the volume V, which has precedent in the variational problems associated with the Yamabe conjecture [38]. We believe that the resulting theory, conformal gravity, is much simpler than any other potentially viable theory of scale-invariant gravity. The use of the volume has the added advantage that V is conserved. This ensures that conformal gravity shares with the particle model the attractive properties it acquires from conservation of the moment of inertia I. We extend this method (of using powers of V to achieve homogeneity) in Secs. 6 and 7 to include matter coupled to conformal gravity. The consequences of this are discussed in Sec. 8. It is the use of V that necessitates our assumption of a spatially compact manifold $\mathcal{M}$. It is a physical assumption, not a mere mathematical convenience. It would not be necessary in the case of theories of the type considered in the previous paragraph.

We must now find and check the consistency of the equations of conformal gravity. The treatment of best matching in the particle model in Sec. PD.4 tells us that we must calculate the canonical momenta of g_{ij} and ϕ, find the conditions that ensure vanishing of the variation of the Lagrangian separately w.r.t to possible independent variations of the auxiliary ϕ and its velocity $v_\phi = \mathrm{d}\phi/\mathrm{d}\lambda$, and then show that these conditions, which involve the canonical momenta, together with the Euler–Lagrange equations for g_{ij} form a consistent set. This implements best matching by the free-end-point method (Sec. PD.4).

The canonical momentum p_ϕ of ϕ is

$$p_\phi \equiv \frac{\partial \mathcal{L}}{\partial v_\phi} = \frac{\sqrt{g}}{2NV^{\frac{2}{3}}} G_{\mathrm{W}}^{abcd} \frac{\mathrm{d}(\phi^4 g_{ab})}{\mathrm{d}\lambda} \frac{4}{\phi} g_{cd}, \tag{34}$$

where $2N = \sqrt{\frac{T_{\mathrm{W}}^{\mathrm{C}}}{R - \frac{8\nabla^2\phi}{\phi}}}$ is (twice) the 'lapse' of matter-free conformal gravity. The gravitational canonical momenta are

$$p^{ab} = \frac{\sqrt{g}}{2NV^{\frac{2}{3}}} G_{\mathrm{W}}^{abcd} \frac{\mathrm{d}(\phi^4 g_{cd})}{\mathrm{d}\lambda}, \tag{35}$$

[9] This is the single symmetry property that distinguishes conformal gravity that distinguishes conformal gravity from GR, which for this reason *just* fails to be fully conformally invariant.

and we see that the canonical momenta satisfy the primary constraint

$$p = \frac{\phi}{4} p_\phi, \tag{36}$$

where

$$p = p^{ab} g_{ab} \tag{37}$$

is the trace of p^{ab} and is the 'localized' analogue of the dilatational momentum (Eq. PD.3). The primary constraint (36) is a direct consequence of the invariance of the action (31) under the banal transformations (28).

Independent variations $\delta\phi$ and δv_ϕ of ϕ and its velocity in the instantaneous Lagrangian that can be considered are: 1) $\delta\phi$ is a spatial constant and $\delta v_\phi \equiv 0$; 2) $\delta\phi$ is a general function of position and $\delta v_\phi \equiv 0$; 3) $\delta\phi \equiv 0$ and $\delta v_\phi \neq 0$ in an infinitesimal spatial region. The possibility 1) has already been used to fix the homogeneity of $\mathcal{L}$. Let us next consider 3). This tells us that $\partial\mathcal{L}/\partial v_\phi = 0$. But $\partial\mathcal{L}/\partial v_\phi \equiv p_\phi$, so we see that the canonical momentum of ϕ must vanish. Then the primary constraint (36) shows that

$$p = 0. \tag{38}$$

As a result of this, without loss of generality, we can set W, the coefficient of the trace, equal to zero, $W = 0$, in the generalization of the DeWitt supermetric used in (31). If conformal gravity proves to be a viable theory, this result could be significant, especially for the quantization programme (see Sec. 8), since it ensures that conformal gravity, in contrast to GR, has a positive-definite kinetic energy. Indeed, in GR, since

$$\frac{\partial g_{ab}}{\partial\lambda} = \frac{2N}{\sqrt{g}}\left(p_{ab} - \frac{p}{2} g_{ab}\right) + 2\nabla_{(a}\xi_{b)}$$

the rate of change of $\sqrt{g}$, which defines the volume element $\sqrt{g}\mathrm{d}^3x$, is measured by the trace p,

$$\frac{\partial\sqrt{g}}{\partial\lambda} = -\frac{Np}{2} + \sqrt{g}\nabla_a\xi^a.$$

Therefore, in conformal gravity the volume element – and with it the volume of 3-space – does not change and cannot make a contribution to the kinetic energy of the opposite sign to the contribution of the shape degrees of freedom. Finally, we consider 2) and (including the use of $p_\phi = 0$) we obtain

$$\phi^3 N\left(R - \frac{7\nabla^2\phi}{\phi}\right) - \nabla^2(\phi^3 N) = \phi^5\left\langle \phi^4 N\left(R - \frac{8\nabla^2\phi}{\phi}\right)\right\rangle, \tag{39}$$

where we use the usual notion of global average:

$$< A >= \frac{\int \mathrm{d}^3x\sqrt{g}A}{\int \mathrm{d}^3x\sqrt{g}}. \tag{40}$$

This lapse-fixing equation holds at all times. Thus it has a status different from the constraints; it has no analogue in the particle model in PD or any other gauge theory of which we are aware. It is, however, a direct consequence of conformal best matching, and as explained below, it plays an important role in the mathematical structure of conformal gravity.

Besides the trace constraint (38), p^{ab} must satisfy the primary quadratic constraint

$$-\mathcal{H}^{\mathrm{C}} \equiv \frac{\sqrt{g}\phi^4}{V^{\frac{2}{3}}}\left(R - \frac{8\nabla^2\phi}{\phi}\right) - \frac{V^{\frac{2}{3}}}{\sqrt{g}\phi^4} p^{ab} p_{ab} = 0 \tag{41}$$

due to the local square-root form of the Lagrangian, and the secondary linear constraint

$$\nabla_b p^{ab} = 0 \tag{42}$$

from variation with respect to ξ_i. Of course, (42) is identical to the GR momentum constraint (3), while (41) is very similar to the Hamiltonian constraint (2).

The Euler–Lagrange equations for g_{ij} are

$$\begin{aligned}\frac{\mathrm{d}p^{ab}}{\mathrm{d}\lambda} = & \frac{\phi^4\sqrt{g}N}{V^{\frac{2}{3}}}\left(Rg^{ab} - \frac{4\nabla^2\phi}{\phi} - R^{ab}\right) - \frac{\sqrt{g}}{V^{\frac{2}{3}}}(g^{ab}\nabla^2(\phi^4 N) - \nabla^a\nabla^b(\phi^4 N)) - \frac{2NV^{\frac{2}{3}}}{\sqrt{g}\phi^4}p^{ac}p^b{}_c \\ & + \frac{8\sqrt{g}}{V^{\frac{2}{3}}}\left(\frac{1}{2}g^{ab}g^{cd} - g^{ac}g^{bd}\right)(\phi^3 N)_{,c}\phi_{,d} - \frac{2\sqrt{g}}{3V^{\frac{2}{3}}}\phi^6 g^{ab}\left\langle \phi^4 N\left(R - \frac{8\nabla^2\phi}{\phi}\right)\right\rangle + \frac{4}{\phi}p^{ab}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}.\end{aligned} \tag{43}$$

They can be used to check the consistency of the full set of equations, constraints and lapse-fixing equation of conformal gravity. To simplify these calculations and simultaneously establish the connection with GR, we go over to the distinguished representation (Sec. PD.4), in which $\phi = 1$ and $\xi^i = 0$. The three constraints that must be satisfied by the gravitational canonical momenta are

$$-\mathcal{H}^{\mathrm{C}} \equiv \frac{\sqrt{g}}{V^{\frac{2}{3}}}R - \frac{V^{\frac{2}{3}}}{\sqrt{g}}p^{ab}p_{ab} = 0, \qquad \nabla_b p^{ab} = 0, \qquad p = 0. \tag{44}$$

The lapse-fixing equation (39) becomes[10]

$$\nabla^2 N - NR = - < NR >, \tag{45}$$

and the Euler–Lagrange equations are

$$\frac{\mathrm{d}p^{ab}}{\mathrm{d}\lambda} = \frac{\sqrt{g}N}{V^{\frac{2}{3}}}(Rg^{ab} - R^{ab}) - \frac{\sqrt{g}}{V^{\frac{2}{3}}}(g^{ab}\nabla^2 N - \nabla^a\nabla^b N) - \frac{2NV^{\frac{2}{3}}}{\sqrt{g}}p^{ac}p^b{}_c - \frac{2\sqrt{g}}{3V^{\frac{2}{3}}}g^{ab} < NR > . \tag{46}$$

Since the volume V is conserved, and its numerical value depends on a nominal length scale, for the purpose of comparison with the equations of GR we can set $V = 1$. (This cannot, of course, be done before the variation that leads to the above equations, since the variation of V generates forces. It is important not to confuse quantities on-shell and off-shell.) We see, setting $V = 1$, that the similarity with GR in York's CMC slicing is strong. In fact, the constraints (44) are identical to the GR constraints and York's slicing condition at maximal expansion, and the Euler–Lagrange equations differ only by the absence of the GR term proportional to p and by the presence of the final force term. This has the same form as the force due to Einstein's cosmological constant, but its strength is fixed by the theory, just as happens for the strength (Eq. PD.39) of the induced cosmological force in the particle model. Finally, there is the lapse-fixing equation (45), which is an eigenvalue equation of essentially the same kind as the lapse-fixing equation (22) required to maintain York's CMC slicing condition (19). In accordance with the last footnote, the left-hand sides are identical, and in both cases the spatial constant on the right-hand side is determined by the functions of position on the left-hand side.

In our view, one of the most interesting results of this work is the derivation of such lapse-fixing equations directly from a fundamental symmetry requirement rather than as an equation which could be interpreted as maintaining a gauge fixing. We now develop this point.

This is part of the confirmation that we do have a consistent set of equations, constraints and lapse-fixing equation. We show this in Sec. 4, which amounts to demonstrating that if the constraints (44) hold initially, then they will propagate due to the Euler–Lagrange equations (46)

[10] We can replace R by $\frac{V^{\frac{4}{3}}}{g}p^{ab}p_{ab}$ in this expression by use of the Hamiltonian constraint, thus bringing the appearance of (45) in accord with (22) and (24). The form (45) is more convenient for later use.

and the lapse-fixing equation (39). The propagation of the vector momentum constraint is always unproblematic, being guaranteed by the 3-diffeomorphism invariance of the theory. The two constraints that could give difficulty are the quadratic and linear scalar constraints. In RWR [11], the propagation of the quadratic constraint proved to be a delicate matter and generated the new results of that paper. However, in this paper we are merely 'conformalizing' the results of [11], and we shall see that the consistency achieved for the quadratic constraint in [11] carries forward to conformal gravity. The only issue is therefore whether the new constraint $p = 0$ is propagated. Now, we find that the form of $\dot{p}$ evaluated from the Euler–Lagrange equations *is automatically guaranteed to be zero* by virtue of the lapse-fixing equation. Thus the propagation of $p = 0$ is guaranteed rather than separately imposed. It is in this sense that conformal gravity is not maintaining a gauge fixing.

We conclude this section with a brief discussion of the thin-sandwich problem (TSP) for conformal gravity. As we said at the start of Sec. 2, the TSP is a serious issue for GR in the Lagrangian formalism. We have a different, not necessarily easier TSP to investigate in conformal gravity. In both cases, we can bypass the TSP by working instead in the Hamiltonian formalism (Sec. 4). It is however worth stating the problem, counting the freedoms and the conditions that should fix them, and making some comments.

In the *conformal thin-sandwich problem*, one specifies g_{ij} and $\mathrm{d}g_{ij}/\mathrm{d}\lambda$ (12 numbers per space point) and uses the momentum constraint, the trace constraint and the lapse-fixing equation (all expressed in Lagrangian variables) to find ϕ and ξ^i that make p_{ij} TT w.r.t the best-matched corrected coordinate $\phi^4 g_{ij}$, where ϕ is part of the solution of the TSP. To solve this problem we have five equations. This is the correct number that we need; four to impose a TT condition just as in York's Hamiltonian method and the fifth to move g_{ij} along its orbit to the best-matched position, which reflects the action of the Lichnerowicz equation (18).[11] The count of freedoms works out too. Among the 12 numbers per space point initially specified, two determine a position in CS and two a direction in CS. These are the physical degrees of freedom, and they are unchanged by the solution of the TSP. Three numbers must remain free, since they correspond to the freedom to specify the coordinates on the manifold freely. The remaining five are corrected by the five conditions per space point.

4 Hamiltonian Formulation and Alternative Theories

In this section, we examine the Hamiltonian formulation. We exercise a number of options in constructing theories such as GR and conformal gravity. In particular, while we have hitherto considered only compact spaces without boundary, we now also touch on the asymptotically-flat case. We also attempt to incorporate both maximal and CMC conditions. Our methods of doing so necessitate a discussion of whether the lapse-fixing equations are fundamental or a gauge-fixing. When we conformally-correct both $\dot{g}_{ij}$ and g_{ij} (which corresponds below to the use of two auxiliary variables which we later find are related), we find that the lapse-fixing equations are variationally guaranteed. This is the case in conformal gravity. When this occurs, the alternative gauge-fixing interpretation is not available. With the hindsight of the last section, one must apply this full correction to obtain the most complete theories. We build up the Hamiltonian structure toward this completeness.

[11] The best matching for conformal gravity thus differs characteristically from the 3-diffeomorphism best matching, which corrects the 'direction of the velocity' in the same way at any point on the orbit in Riem. In the conformal problem, both the direction of the velocity and the position on the orbit are corrected. Many relativists find the need for and effect of the Lichnerowicz transformation puzzling. We believe this and the next section provide a transparent geometrical (best-matching) explanation of the transformation.

We begin with the case that corresponds to working on superspace. ADM and Dirac showed that the Hamiltonian for GR can be written as (1) – (3):

$$\mathsf{H} = \int \left(N \left(\frac{1}{\sqrt{g}} \left(p^{ab} p_{ab} - \frac{p^2}{2} \right) - \sqrt{g} R \right) + \xi^i \left(-2 \nabla_j p^j{}_i \right) \right) \mathrm{d}^3 x. \tag{47}$$

Hamilton's equations are then the evolution equations

$$\frac{\mathrm{d}g_{ab}}{\mathrm{d}\lambda} = \frac{2N}{\sqrt{g}} \left(p_{ab} - \frac{p}{2} g_{ab} \right), \tag{48}$$

$$\frac{\mathrm{d}p^{ab}}{\mathrm{d}\lambda} = \sqrt{g} N \left(\frac{R}{2} g^{ab} - R^{ab} \right) - \sqrt{g} (g^{ab} \nabla^2 N - \nabla^a \nabla^b N) + \frac{N}{2\sqrt{g}} g^{ab} \left(p^{ij} p_{ij} - \frac{p^2}{2} \right) - \frac{2N}{\sqrt{g}} \left(p^{ac} p^b{}_c - \frac{p}{2} p^{ab} \right). \tag{49}$$

Here, N and ξ^i are arbitrary and the evolution equations propagate the constraints. These are equivalent to the BSW evolution equations of RWR.

Now we wish to work with CMC slices on CS+V. In this case we find that there is little difference between the compact and asymptotically-flat cases. We can treat this in the Hamiltonian framework simply by adding another constraint to the Hamiltonian (47), i.e by considering

$$\mathsf{H}^\eta = \int \mathrm{d}^3 x (N \mathcal{H} + \xi^i \mathcal{H}_i + (\nabla^2 \eta) p) \tag{50}$$

and treating η as another Lagrange multiplier. The Laplacian is introduced here to obtain the CMC condition as the new constraint arising from η-variation: $\nabla^2 p = 0 \Rightarrow p/\sqrt{g} = \text{const.}$. In addition we get the standard GR Hamiltonian and momentum constraints from variation w.r.t the Lagrange multipliers N and ξ^i. Now, we can impose $\frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{p}{\sqrt{g}} \right) = const$ to arrive at the CMC lapse-fixing equation. Whereas one could reinterpret this as the study in the CMC gauge of the subset of (pieces of) GR solutions which are CMC foliable, we can also consider this to be a new theory with a preferred fundamental CMC slicing.[12] The latter interpretation is our first CS+V theory.

This CS+V theory's evolution equations are (48) and (49) but picking up the extra terms $g_{ab} \nabla^2 \eta$ and $-p^{ab} \nabla^2 \eta$ respectively. We have already dealt with the CMC constraint; it turns out that we need to set $\nabla^2 \eta = 0$ to preserve the other constraints. Therefore the CMC Hamiltonian is well-defined when one makes the gauge choices of $\nabla^2 \eta = 0$ and that N satisfies the CMC lapse-fixing equation.

It is more satisfactory however to introduce a second auxiliary variable to conformally correct the objects associated with the metric. We use $(1 + \nabla^2 \zeta)^{\frac{1}{6}}$ in place of ϕ since this implements volume-preserving conformal transformations

$$\bar{V} - V = \int \mathrm{d}^3 x \sqrt{g} \left((1 + \nabla^2 \zeta)^{\frac{1}{6}} \right)^6 - \int \mathrm{d}^3 x \sqrt{g} = \int \mathrm{d}^3 x \sqrt{g} \nabla^2 \zeta = \oint \mathrm{d}S_a \nabla^a \zeta = 0 \tag{51}$$

in the case where the 3-geometry is closed without boundary. Applying these corrections, we obtain the Hamiltonian

$$\mathsf{H}^{\zeta\eta} = \int \mathrm{d}^3 x (N \mathcal{H}^\zeta + \xi^i \mathcal{H}_i + (\nabla^2 \eta) p), \tag{52}$$

$$\mathcal{H}^\zeta \equiv \frac{1}{\sqrt{g}} \left(\frac{\sigma_{ij} \sigma^{ij}}{(1 + \nabla^2 \zeta)^{\frac{2}{3}}} - \frac{(1 + \nabla^2 \zeta)^{\frac{4}{3}} p^2}{6} \right) - \sqrt{g} (1 + \nabla^2 \zeta)^{\frac{2}{3}} \left(R - \frac{8 \nabla^2 (1 + \nabla^2 \zeta)^{\frac{1}{6}}}{(1 + \nabla^2 \zeta)^{\frac{1}{6}}} \right). \tag{53}$$

[12] By a preferred fundamental slicing, we mean a single stack of Riemannian 3-spaces. This is not to be confused with GR, where there are an infinity of such stacks between any two given spacelike hypersurfaces, each of which is called a slicing or foliation of that spacetime region.

Note that the conformal weighting of $\sigma_{ij}\sigma^{ij}$ and p^2 is distinct, just as in York's method! We require p to transform as a conformal scalar. Now, N, ξ^i, η variation yields $\mathcal{H}^\zeta = 0$, $\mathcal{H}_i = 0$, $p/\sqrt{g} = const.$ Then in the distinguished representation $\nabla^2\eta = \nabla^2\zeta = 0$, ζ-variation and one use of $\mathcal{H}^\zeta = 0$ yields

$$NR - \nabla^2 N + \frac{Np^2}{4g} = const. \tag{54}$$

Now, this is of the correct form to automatically guarantee that $\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{p}{\sqrt{g}}\right) = const.$ Thus the CMC lapse-fixing equation is variationally encoded in this *full CS+V theory*. Furthermore, $\mathcal{H}^\zeta = 0$ is the extended Lichnerowicz equation (21) for $\phi = 1 + \nabla^2\zeta$. It is this equation that carries the auxiliary variables which encode the CMC lapse-fixing equation.

We now try to extend the above workings to the case of the maximal condition. This involves starting with a slightly different Hamiltonian,

$$\mathsf{H}^\theta = \int \mathrm{d}^3x(N\mathcal{H} + \xi^i\mathcal{H}_i - \theta p). \tag{55}$$

Variation w.r.t θ gives us the maximal condition $p = 0$. In the asymptotically-flat case, it makes sense to impose $\dot{p} = 0$ to arrive at the maximal lapse-fixing equation

$$\nabla^2 N = RN, \tag{56}$$

since this is an extremely well-behaved equation when taken in conjunction with the boundary condition $N \longrightarrow 1$ at infinity. Again, variation w.r.t the multipliers N, ξ^i yields the standard GR Hamiltonian and momentum constraints respectively. Now if we choose $\theta = 0$ in the evolution we preserve the Hamiltonian and momentum constraints. Therefore H^θ can be interpreted as yielding standard GR with a choice of maximal gauge, or alternatively that this represents a new theory with a preferred fundamental maximal slicing: our first asymptotically-flat theory.

This does not work in the case of a compact manifold without boundary, since now the only solution of (56) is $N \equiv 0$ and so we get frozen dynamics. But if we use the volume of the universe (which amounts to moving away from GR), we are led to new theories. First, consider

$$\mathsf{H}^{\theta\mathrm{V}} = \int \mathrm{d}^3x(N\mathcal{H}^{\mathrm{V}} + \xi^i\mathcal{H}_i - \theta p)\ ,\ \mathcal{H}^{\mathrm{V}} \equiv \frac{V^{\frac{2}{3}}}{\sqrt{g}}p^{ab}p_{ab} - \frac{\sqrt{g}}{V^{\frac{2}{3}}}R \tag{57}$$

Then the N, ξ^i, θ variations yield the constraints $\mathcal{H}^V = 0$, $\mathcal{H}_i = 0$ and $p = 0$. Imposing $\dot{p} = 0$ yields a maximal lapse-fixing equation

$$\nabla^2 N = RN - < RN > . \tag{58}$$

Whereas this could be regarded as a gauge fixing, the underlying theory is no longer GR. The other interpretation is that of a new theory with a preferred fundamental maximal slicing.

Second, consider the use of two auxiliary conformal variables:

$$\mathsf{H}^{\phi\theta\mathrm{V}} = \int \mathrm{d}^3x(N\mathcal{H}^{\mathrm{C}} + \xi^i\mathcal{H}_i - \theta p)\ ,\ \mathcal{H}^{\mathrm{C}} \equiv \frac{V^{\frac{2}{3}}}{\phi^4\sqrt{g}}p^{ab}p_{ab} - \frac{\phi^4\sqrt{g}}{V^{\frac{2}{3}}}\left(R - \frac{8\nabla^2\phi}{\phi}\right). \tag{59}$$

Then the N, ξ^i, θ variations yield the constraints $\mathcal{H}^{\mathrm{C}} = 0$, $\mathcal{H}_i = 0$ and $p = 0$. Hamilton's equations are now the evolution equations

$$\begin{aligned}
\frac{\mathrm{d}g_{ab}}{\mathrm{d}\lambda} &= \frac{2NV^{\frac{2}{3}}}{\sqrt{g}}p_{ab} - \theta g_{ab}, \\
\frac{dp^{ab}}{d\lambda} &= \frac{\sqrt{g}N}{V^{\frac{2}{3}}}\left(\frac{R}{2}g^{ab} - R^{ab}\right) - \frac{\sqrt{g}}{V^{\frac{2}{3}}}(g^{ab}\nabla^2 N - \nabla^a\nabla^b N) \\
&\quad + \frac{NV^{\frac{2}{3}}}{2\sqrt{g}}g^{ab}p^{ij}p_{ij} - \frac{2NV^{\frac{2}{3}}}{\sqrt{g}}p^{ac}p^b{}_c - \frac{2}{3V^{\frac{2}{3}}}\langle NR\rangle g^{ab} + \theta p^{ab}
\end{aligned} \tag{60}$$

in the distinguished representation $\phi = 1$. Whilst ϕ variation yields (58), Hamilton's equations give

$$\dot{p} = \frac{2\sqrt{g}}{V^{\frac{2}{3}}}(NR - < NR > - \nabla^2 N) \tag{61}$$

which is thus automatically satisfied due to (58). This theory is (full) conformal gravity, as can be confirmed by Legendre transformation and BSW elimination [14] to recover the Lagrangian of Sec. 3. A nontrivial step in this procedure of justifying the Hamiltonian (59) to be that of conformal gravity is presented shortly. Because the lapse-fixing equation that maintains the condition is also guaranteed, conformal gravity is definitely *not* interpretable as a gauge fixing.

Before turning to these, we mention that there is one further maximal theory which arises from considering two conformal auxiliary variables in the asymptotically flat case:

$$\mathsf{H}^{\phi\theta} = \int \mathrm{d}^3x(N\mathcal{H}^\phi + \xi^i\mathcal{H}_i - \theta p)\ ,\ \mathcal{H}^\phi \equiv \frac{1}{\phi^4\sqrt{g}}p^{ab}p_{ab} - \phi^4\sqrt{g}\left(R - \frac{8\nabla^2\phi}{\phi}\right). \tag{62}$$

Then N, ξ^i, θ variation yield the constraints $\mathcal{H}^\phi = 0$, $\mathcal{H}_i = 0$ and $p = 0$. Now, ϕ variation gives the maximal lapse-fixing equation (56), and so automatically guarantees the propagation of the condition $p = 0$. This is our *full asymptotically flat theory.* Note that $\mathcal{H}^\phi = 0$ is the Lichnerowicz equation (18). Again, the Lichnerowicz equation carries the auxiliary variable which encode the lapse-fixing equation. For conformal gravity itself, we see it as significant that one must modify the corresponding Lichnerowicz equation by the introduction of volume terms to get an analogous scheme.

Like conformal gravity, our full asymptotically-flat theory has no role for p in its dynamics, but, unlike conformal gravity, it does not possess global terms. We are less interested in this theory than in conformal gravity because it would not be immediately applicable to cosmology, on account of being asymptotically flat. These two theories should be contrasted with our full CS+V theory, in which p does play a role, which means that the standard GR explanation of cosmology is available. We believe the CS+V theory merits a full treatment elsewhere as another potential rival to GR.

The theories (50), (55, (57) above may be viewed as a poor man's versions of our three full theories. The formulation provided by their gauge interpretation is new and may be useful in numerical relativity.

As in the Appendix of PD, we now justify the Hamiltonians of GR and conformal gravity (which are the main theories discussed in this paper) from the point of view of free-end-point variation. Lanczos [3] treats the Hamiltonian method as a special case of the Lagrangian method for which the kinetic energy has the form $T = \Sigma_i p^i\dot{q}_i$. Thus we treat the momenta p^i as coordinates on an equal footing with the q_i. Constraints are to be appended with Lagrange multipliers. We furthermore treat all the variables that we regard as gauge auxiliaries by the free-end-point method.

For GR, this includes treating the shift as the velocity of some auxiliary variable f^i: $\xi^i = \dot{f}^i$. Thus

$$L = \int \mathrm{d}^3x\mathcal{L} = \int \mathrm{d}^3x(p^{ij}\dot{g}_{ij} + p^{\mathrm{f}}_i\dot{f}^i - N\mathcal{H} - S^i(p^{\mathrm{f}}_i - 2\nabla_j p^j{}_i)), \tag{63}$$

there is by construction no f^i variation and $\dot{f}^i$ variation yields $p^{\mathrm{f}}_i = 0$. Variation w.r.t the true multipliers N and S_i yields respectively the Hamiltonian constraint $\mathcal{H} = 0$ and $-2\nabla_j p^j{}_i = -p^{\mathrm{f}}_i$ (i.e the momentum constraint $\mathcal{H}_i = 0$). Variation w.r.t p^{f}_i yields $S^i = \dot{f}^i = \xi^i$, thus recovering the habitual Lagrange multiplier notion for the shift, and variation w.r.t p^{ij} and g_{ij} yields the usual ADM evolution equations. Thus we can write $L = \int \mathrm{d}^3x(p^{ij}\dot{g}_{ij} - N\mathcal{H} - \xi^i\mathcal{H}_i)$, so it is consistent for the Hamiltonian to take its usual GR form, $\mathsf{H} = \int \mathrm{d}^3x(N\mathcal{H} + \xi^i\mathcal{H}_i)$.

For conformal gravity,

$$L=\int \mathrm{d}^3x\left(p^{ij}\dot{g}_{ij}+p^{\mathrm{f}}_i\dot{f}^i+p_\phi\dot{\phi}-N\mathcal{H}^{\mathrm{C}}-S^i(p^{\mathrm{f}}_i-2\nabla_j p^j{}_i)-\theta\left(\frac{\phi}{4}p_\phi-p^{ij}\dot{g}_{ij}\right)\right), \tag{64}$$

Repeat all the GR steps (obtaining $\mathcal{H}^{\mathrm{C}}=0$ and the conformal gravity evolution equations in place of their GR ADM counterparts). In addition we have a new auxiliary variable ϕ; variation w.r.t this yields the lapse-fixing equation (58) and $\dot{\phi}$ variation yields $p_\phi=0$. We also have a new true multiplier θ variation w.r.t which yields $p=\frac{\phi}{4}p_\phi=0$. The virtue of (64) is that p_ϕ variation directly yields $\theta=4\frac{\dot{\phi}}{\phi}$. Thus we can write $L=\int \mathrm{d}^3x(p^{ij}\dot{g}_{ij}-N\mathcal{H}^{\mathrm{C}}-\xi^i\mathcal{H}_i+\theta p)$, so it is consistent to take the conformal gravity Hamiltonian to be $\int \mathrm{d}^3x(N\mathcal{H}^{\mathrm{C}}+\xi^i\mathcal{H}_i-\theta p)$.

We now consider the preservation of the other conformal gravity constraints $\mathcal{H}^{\mathrm{C}}$ and $\mathcal{H}_i$. We find that we need to set $\theta=0$. Therefore (by comparison with GR) the only term we need to worry about is the $-\frac{2}{3V^{\frac{2}{3}}}<NR>g^{ab}$ term in the $\dot{p}^{ab}$ equation. Since it is of the form Cg^{ab} it clearly will not disturb the momentum constraint. Therefore we need only worry about conserving the Hamiltonian constraint. This is quite straightforward. We first realize that

$$\frac{\partial\sqrt{g}}{\partial\lambda}=\sqrt{g}\nabla_i\xi^i. \tag{65}$$

Hence

$$\frac{\partial V}{\partial\lambda}=0. \tag{66}$$

The only other term to worry about is $\frac{V^{\frac{2}{3}}}{\sqrt{g}}p_{ab}p^{ab}$. Varying p^{ab} gives $-\frac{4}{3\sqrt{g}}<NR>p=0$. Therefore the constraints are preserved under evolution.

We now show that we can just as easily treat the Hamiltonian dynamics of conformal gravity in the general representation. The lapse-fixing equation from ϕ variation is now

$$\nabla^2(\phi^3N)=\phi^3N\left(R-\frac{7\nabla^2\phi}{\phi}\right)-\phi^5\left\langle\phi^4N\left(R-\frac{8\nabla^2\phi}{\phi}\right)\right\rangle, \tag{67}$$

whilst Hamilton's evolution equations are now

$$\begin{aligned}\frac{\mathrm{d}g_{ab}}{\mathrm{d}\lambda}&=\frac{2NV^{\frac{2}{3}}}{\phi^4\sqrt{g}}p_{ab}-\theta g_{ab},\\ \frac{dp^{ab}}{d\lambda}&=\frac{\phi^4\sqrt{g}N}{V^{\frac{2}{3}}}\left(\frac{R}{2}g^{ab}-R^{ab}\right)+\frac{NV^{\frac{2}{3}}}{2\phi^4\sqrt{g}}p^{cd}p_{cd}g^{ab}-\frac{\sqrt{g}}{V^{\frac{2}{3}}}(g^{ab}\nabla^2(\phi^4N)-\nabla^a\nabla^b(\phi^4N))+\theta p^{ab}\\ &-\frac{2NV^{\frac{2}{3}}}{\sqrt{g}\phi^4}p^{ac}p^b{}_c+\frac{8\sqrt{g}}{V^{\frac{2}{3}}}\left(\frac{1}{2}g^{ab}g^{cd}-g^{ac}g^{bd}\right)(\phi^3N)_{,c}\phi_{,d}-\frac{2\sqrt{g}}{3V^{\frac{2}{3}}}\phi^6g^{ab}\left\langle\phi^4N\left(R-\frac{8\nabla^2\phi}{\phi}\right)\right\rangle.\end{aligned} \tag{68}$$

We can compare these expressions to their Lagrangian analogues (35), (43) and we see they coincide if $\theta=\frac{4}{\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}$. This will guarantee that the constraints are preserved by the evolution. Alternatively, we could evolve the constraints using the Hamiltonian evolution equations (68) and discover that the constraints propagate if and only if the lapse function, N, satisfies (67), the shift, ξ, is arbitrary, and θ satisfies $\theta=\frac{4}{\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}$. The ϕ variation gives the lapse-fixing equation (39). We emphasize that $\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}$ is arbitrary in the full theories, unlike in the poor man's versions, where one ends up having to set the auxiliary (θ or $\nabla^2\eta$) to zero.

It is not obvious then that $\dot{p}=0$ is guaranteed form Hamilton's equations, since what one immediately obtains is, weakly,[13]

$$\frac{\partial p}{\partial\lambda}\approx\frac{2N\sqrt{g}\phi^4}{V^{\frac{2}{3}}}\left(R-\frac{6\nabla^2\phi}{\phi}\right)-\frac{2\sqrt{g}}{V^{\frac{2}{3}}}\nabla^2(\phi^4N)+\frac{4\sqrt{g}}{V^{\frac{2}{3}}}g^{cd}(\phi^3N)_{,c}\phi_{,d}-\frac{2\sqrt{g}}{V^{\frac{2}{3}}}\phi^6\left\langle\phi^4N\left(R-\frac{8\nabla^2\phi}{\phi}\right)\right\rangle. \tag{69}$$

[13] We use $\approx$ to denote weak vanishing as defined by Dirac [4], i.e equality modulo constraints.

We now require use of $\nabla^2(\phi^4 N) = \phi\nabla^2(\phi^3 N) + 2g^{cd}(\phi^3 N)_{,c}\phi_{,d} + \phi^3 N\nabla^2\phi$ to see that (39) indeed guarantees $\dot{p} = 0$.

A different application of the Dirac procedure used in this section is given for relativity in [39]. One starts with the BSW action (5), but omits the best-matching 3-diffeomorphism corrections to the bare velocities in the kinetic term. There is no ξ^i to vary in order to obtain the momentum constraint. However, if one again uses the evolution equations to propagate the quadratic square-root constraint, the momentum constraint arises as a necessary integrability condition. This formalism is of interest in establishing the extension of the 3-space approach to include fermionic matter [40]. These issues in conformal gravity are beyond the scope of this paper and will be the subject of further work.

Finally, we briefly mention two more sources of variety in our family of theories. First, in RWR [11], the most general consistent BSW-type pure gravity action considered is

$$I = \int \mathrm{d}\lambda\sqrt{g}\sqrt{\Lambda + sR}\sqrt{T_{\mathrm{W}}}\,. \tag{70}$$

For $s = 1, -1$ this gives $W = 1$ Lorentzian and Euclidean GR respectively, whilst for $s = 0$ it gives *strong gravity* [15] generalized to arbitrary W [16]. We find that there is analogously a *strong conformal theory* (which this time has without loss of generality $W = 0$), with action

$$S^{\mathrm{Strong}} = \int \mathrm{d}\lambda\frac{\int \mathrm{d}^3x\sqrt{g}\phi^6\sqrt{\Lambda}\sqrt{T^{\mathrm{C}}}}{V}, \tag{71}$$

which may be of use in understanding quantum conformal gravity. Note that the power of the volume V needed to make the action homogeneous of degree zero is here one, since now it has to balance only $\sqrt{g}$ and not the product $\sqrt{g}\sqrt{R}$. This theory is simpler than conformal gravity in two ways: Λ is less intricate than R, and the lapse-fixing equation for strong conformal gravity is $\Lambda N =< \Lambda N >$ so since $< \Lambda N >$ is a spatial constant, N is a spatial constant.

Second, instead of constructing an action with a local square root, one could use instead a *global* square root and thus obtain

$$S^{\mathrm{Global}} = \int \mathrm{d}\lambda\frac{\sqrt{\int \mathrm{d}^3x\sqrt{g}\phi^2\left(R - \frac{8\nabla^2\phi}{\phi}\right)}\sqrt{\int \mathrm{d}^3x\sqrt{g}\phi^6 T^{\mathrm{C}}}}{V^{\frac{2}{3}}}, \tag{72}$$

which gives rise to a single global quadratic constraint. We note that the above alternatives are cumulative: for each of the theories in this section, we could consider 6 variants by picking Euclidean, strong, or Lorentzian signature and a local or global square root. The Lorentzian, local choices expanded above are the most obviously physical choice.

5 Integral Conditions and the Cosmological Constant

We first demonstrate for pure conformal gravity that the $\frac{2}{3}$ power that makes the action homogeneous is indeed required. For, if we change the action to

$$S^{C_n} = \int \mathrm{d}\lambda\frac{\int \mathrm{d}^3x\sqrt{g}\phi^4\sqrt{R - \frac{8\nabla^2\phi}{\phi}}\sqrt{T^{\mathrm{C}}}}{V^n} \tag{73}$$

with n a free power, we get the modified lapse-fixing equation (45)

$$RN - \nabla^2 N = \frac{3}{2}n < NR > \tag{74}$$

in the distinguished representation. Then integration of (74) over space yields

$$0 = \oint \sqrt{g}\mathrm{d}S_a \nabla^a N = \int \mathrm{d}^3x\sqrt{g}\nabla^2 N = \int \mathrm{d}^3x\sqrt{g}RN - \frac{3}{2}n\int \mathrm{d}^3x\sqrt{g(x)}\left(\frac{\int \mathrm{d}^3y\sqrt{g(y)}R(y)N(y)}{V}\right)$$

$$= \left(1-\frac{3}{2}n\right)\int \mathrm{d}^3x\sqrt{g}RN, \tag{75}$$

using, respectively, the fact that the manifold is compact without boundary, Gauss's theorem, (74), and that spatial integrals are constants and can therefore be pulled outside further spatial integrals. This shows that consistency requires $n = 2/3$, since the integral cannot vanish. Likewise strong conformal gravity requires the power of the volume to be 1.

We recall that the above powers of the volume were found by requiring the actions to be invariant under the constant rescaling of ϕ. Thus, the integral consistency check (75) indeed shows that this homogeneity invariance is indispensable for the production of consistent actions in the compact case without boundary. If it is not observed, one obtains pathological frozen dynamics: $N \equiv 0$. We note that in contrast, in the asymptotically-flat case considered in Sec. 4, we need not divide by a volume term in the action, and the integral inconsistency argument is not applicable.

We now ask what happens when one attempts to combine the actions of strong conformal gravity and conformal gravity in order to consider conformal gravity with a cosmological constant. Applying the homogeneity requirement, we obtain the combined action

$${}^{\Lambda}S = \int \mathrm{d}\lambda \int \mathrm{d}^3x\left(\left(\frac{\sqrt{g}\phi^6}{V}\right)\sqrt{\left(\frac{V^{\frac{2}{3}}}{\phi^4}\right)s\left(R-\frac{8\nabla^2\phi}{\phi}\right)+\Lambda}\sqrt{T^{\mathrm{C}}}\right)$$

$$= \int \mathrm{d}\lambda\frac{\int \mathrm{d}^3x\sqrt{g}\phi^4\sqrt{s\left(R-\frac{8\nabla^2\phi}{\phi}\right)+\frac{\Lambda\phi^4}{V(\phi)^{\frac{2}{3}}}}\sqrt{T^{\mathrm{C}}}}{V(\phi)^{\frac{2}{3}}} = \int \mathrm{d}\lambda\frac{\bar{J}}{V^{\frac{2}{3}}}, \tag{76}$$

with cosmological constant Λ, where we have also included a signature s to show how (76) reduces to the strong conformal gravity action (71) in the limit $s \longrightarrow 0$.

The conjugate momenta p^{ij} and p_ϕ are given by (34) and (35) as before, but now with

$$2N = \sqrt{\frac{T^{\mathrm{C}}}{s\left(R-\frac{8\nabla^2\phi}{\phi}\right)+\frac{\Lambda\phi^4}{V(\phi)^{\frac{2}{3}}}}},$$

and the primary constraint (36) holds. Again, the end-point part of the ϕ variation yields $p_\phi = 0$, so $p = 0$, so without loss of generality $W = 0$, but now the rest of the ϕ variation gives a new lapse-fixing equation,

$$2s(NR - \nabla^2 N) + \frac{3N\Lambda}{V^{\frac{2}{3}}} = \frac{\bar{J}}{V} + \frac{< N\Lambda >}{V^{\frac{2}{3}}} \tag{77}$$

in the distinguished representation. For this choice of the action, there is indeed no integral inconsistency:

$$0 = 2s\oint \sqrt{g}\mathrm{d}S_a\nabla^a N = 2s\int \mathrm{d}^3x\sqrt{g}\nabla^2 N = \int \mathrm{d}^3x\sqrt{g}\left(2sRN + \frac{3N\Lambda}{V^{\frac{2}{3}}} - 2N\left(sR+\frac{\Lambda}{V^{\frac{2}{3}}}\right) - \frac{N\Lambda}{V^{\frac{2}{3}}}\right). \tag{78}$$

The ξ^i-variation yields the usual momentum constraint (3), and the local square root gives the constraint

$$-{}^{\Lambda}\mathcal{H}^{\mathrm{C}} \equiv \frac{\sqrt{g}}{V^{\frac{2}{3}}}\left(sR+\frac{\Lambda}{V^{\frac{2}{3}}}\right) - \frac{V^{\frac{2}{3}}}{\sqrt{g}}g_{ik}g_{jl}p^{ij}p^{kl} = 0. \tag{79}$$

in the distinguished representation. The Euler–Lagrange equations are

$$\frac{\mathrm{d}p^{ij}}{\mathrm{d}\lambda} = \frac{s\sqrt{g}N}{V^{\frac{2}{3}}}(g^{ij}R - R^{ij}) + \frac{\sqrt{g}s}{V^{\frac{2}{3}}}(\nabla^i\nabla^j N - g^{ij}\nabla^2 N) - \frac{2NV^{\frac{2}{3}}}{\sqrt{g}}p^{im}p^j{}_m - \frac{\bar{J}\sqrt{g}}{V^{\frac{5}{3}}}g^{ij} + \frac{\Lambda\sqrt{g}}{V^{\frac{4}{3}}}g^{ij}\left(N - \frac{<N>}{3}\right), \tag{80}$$

where we have split the working up into pure conformal gravity and Λ parts (the $\bar{J}$ here can also be split into the pure conformal gravity integrand and a Λ part).

Then by (77) $\dot{p} \approx 0$, and finally

$$-{}^{\Lambda}\dot{\mathcal{H}}^{\mathrm{C}} \approx \frac{s\sqrt{g}}{V^{\frac{2}{3}}}\frac{\partial R}{\partial\lambda} - \frac{2V^{\frac{2}{3}}}{\sqrt{g}}\left(\frac{\partial p^{ij}}{\partial\lambda}p_{ij} + \frac{\partial g_{ik}}{\partial\lambda}p^{ij}p^k{}_j\right) \approx 0 \tag{81}$$

by the use of (65) and (66) in the first step, and in the second step $\frac{\partial p^{ij}}{\partial\lambda} = \frac{\partial p^{ij}}{\partial\lambda}|_{\Lambda\text{-free}} + \frac{\partial p^{ij}}{\partial\lambda}|_{\Lambda}$ and $\frac{\partial p^{ij}}{\partial\lambda}|_{\Lambda} \propto g^{ij}$, so p_{ij} annihilates this term, thus reducing ${}^{\Lambda}\dot{\mathcal{H}}^{\mathrm{C}}$ to the pure conformal gravity $\dot{\mathcal{H}}^{\mathrm{C}} \approx 0$.

Note that in conformal gravity the cosmological constant Λ (just like its particle model analogue, the Newtonian energy E) contributes to a conformally-induced cosmological-constant type force. The penultimate term in (80) is the final term of pure conformal gravity in (46), and the final term is induced by Λ. Next, we will see that matter also gives analogous contributions. The significance of this is discussed in Sec. 8.

6 Conformal Gravity Coupled to Matter Fields: General Results

It is an important test of our theoretical framework to see whether it is capable of accommodating enough classical field theories to be a viable description of nature. In RWR [11, 12], we showed that if matter fields are 'added on', GR imposes a universal light cone and requires 3-vector fields to be gauge fields. We now show below how these results carry over to conformal gravity.

In conformal gravity, we have to check the propagation of three different constraints. The momentum constraint $\mathcal{H}_i$ is never problematic, but the other two – the quadratic constraint $\mathcal{H}^{\mathrm{C}}$ that arises from the local square-root form of the BSW action and the linear conformal constraint $p = 0$ – need to be carefully checked. The momentum and quadratic square-root constraints were shown in RWR [11] to create the Efe's, the universal light cone obeyed by the bosonic matter and gauge theory. This paper shows how the matter results also hold in conformal gravity where the linear conformal constraint is also present.

We begin with two theorems that suffice for the construction of a range of classical field theories coupled to conformal gravity. The first is about homogeneity and the propagation of $p = 0$ and the second is about the propagation of $\mathcal{H}^{\mathrm{C}}$. We will then demonstrate example by example that the range of theories covered by these theorems includes all of known classical bosonic physics coupled to conformal gravity. Furthermore, these theories are singled out from more general possibilities by exhaustive implementation of Dirac's demand for dynamical consistency.

Let Ψ be a set of matter fields that we wish to couple to conformal gravity, with potential term ${}^{\Psi}U$ and kinetic term ${}^{\Psi}T$. We first decompose these as polynomials in the inverse metric. This is because it is the power of the metric that determines the powers of V that must be used to achieve the necessary homogeneity. Let $\Phi^{(n)}$ be the set of fields such that these polynomials are of no higher degree than n. Thus
${}^{\Psi^{(n)}}T = \Sigma^{(n)}_{(k=0)}T^{(k)}_{i_1j_1i_2j_2\ldots i_kj_k}g^{i_1j_1}\ldots g^{i_kj_k} = \Sigma^{(n)}_{(k=0)}T_{(k)}$, ${}^{\Psi^{(n)}}U = \Sigma^{(n)}_{(k=0)}U^{(k)}_{i_1j_1i_2j_2\ldots i_kj_k}g^{i_1j_1}\ldots g^{i_kj_k} = \Sigma^{(n)}_{(k=0)}U_{(k)}$.

Then the following theorem guarantees that $p = 0$ is preserved by the dynamical evolution.

Theorem 1
For matter fields $\Psi^{(n)}$, the conformal gravity + matter action of the form

$$
{}^{\Psi^{(n)}}S = \int \mathrm{d}\lambda \frac{\int \mathrm{d}^3x\sqrt{g}\phi^4\sqrt{s\left(R - \frac{8\nabla^2\phi}{\phi}\right) + \frac{\phi^4}{V^{\frac{2}{3}}}\Sigma^{(n)}_{(k=0)}\frac{U_{(k)}V^{\frac{2k}{3}}}{\phi^{4k}}}\sqrt{T^{\mathrm{C}} + \Sigma^{(n)}_{(k=0)}\frac{T_{(k)}V^{\frac{2k}{3}}}{\phi^{4k}}}}{V^{\frac{2}{3}}} \tag{82}
$$

varied with free end points is guaranteed to have $\dot{p} = 0 \ \forall \ n \in \mathcal{N}_0$.

Note how the powers of V match the powers of the inverse metric that are needed to make 3-diffeomorphism scalars from the matter fields of different possible ranks.
Proof Vacuum conformal gravity holds, hence the theorem is true for $n = 0$.
Induction hypothesis: suppose the theorem is true for some $n = q$.
Then, for $n = q + 1$, ϕ variation gives

$$
0 = \frac{\delta S^{(q+1)}}{\delta\phi(x)} = \frac{\delta S^{(q)}}{\delta\phi(x)} \begin{array}{l} +4(2-q)NV^{\frac{2(q-1)}{3}}U_{(q+1)} - \frac{q+1}{N}T_{(q+1)}V^{\frac{2}{3q}} \\ +4\int \mathrm{d}^3x\sqrt{g}V^{\frac{2q-5}{3}}\left(qNU_{(q+1)} + \frac{q+1}{4N}V^{\frac{2}{3}}T_{(q+1)}\right). \end{array} \tag{83}
$$

Now, from $\dot{p}^{(q+1)} = \dot{p}^{(q+1)ij}g_{ij} + p^{(q+1)ij}\dot{g}_{ij}$ and the metric Euler–Lagrange equation for $\dot{p}^{(q+1)ij}$,

$$
\begin{aligned}
\dot{p}^{(q+1)} = \ & \dot{p}^{(q)} + \int \mathrm{d}^3x\sqrt{g}V^{\frac{2q-5}{3}}\left(NU_{(q+1)} + \frac{V^{\frac{2}{3}}}{4N}T_{(q+1)}\right) + 3N\sqrt{g}U_{(q+1)}V^{\frac{2(q-2)}{3}} \\
& + V^{\frac{2(q-2)}{3}}\sqrt{g}\left(\frac{V^{\frac{2}{3}}}{4N}\frac{\delta T_{(q+1)}}{\delta g_{ij}} + N\frac{\delta U_{(q+1)}}{\delta g_{ij}}\right)g^{ij}
\end{aligned} \tag{84}
$$

Hence, by (83)

$$
\dot{p}^{(q+1)} = V^{\frac{2(q-2)}{3}}\sqrt{g}\left(\frac{V^{\frac{2}{3}}}{4N}\left(\frac{\delta T_{(q+1)}}{\delta g_{ij}}g^{ij} + (q+1)T_{(q+1)}\right) + N\left(\frac{\delta U_{(q+1)}}{\delta g_{ij}}g^{ij} + (q+1)U_{(q+1)}\right)\right) = 0 \tag{85}
$$

by the induction hypothesis and using that $U_{(q+1)}$, $T_{(q+1)}$ are homogeneous of degree $q + 1$ in g^{ij}. Hence, if the theorem is true for $n = q$, it is also true for $n = q + 1$. But it is true for $n = 0$, so it is true by induction $\forall \ n \in \mathcal{N}_0$. $\Box$

We will now consider ${}^{\Psi}T$ and ${}^{\Psi}U$ as being made up of contributions from each of the fields present. We will label these fields, and the indices they carry, by capital Greek indexing sets. We then obtain the following formulae for the propagation of the Hamiltonian constraint.
Theorem 2
i) For nonderivative coupled matter fields Ψ_Δ with ${}^{\Psi}T$ homogeneously quadratic in $\dot{\Psi}_\Delta$ and ${}^{\Psi}U$ containing at most first-order derivatives,

$$
-{}^{\Psi}\dot{\mathcal{H}}^{\mathrm{C}} = \frac{1}{N}\nabla_b\left(N^2\left(2G_{\Delta\Gamma}\Pi^{\Gamma}\frac{\partial^{\Psi}U}{\partial(\nabla_b\Psi_\Delta)} + s\left[\Pi^{\Gamma}\frac{\delta(\pounds_\xi\Psi_\Gamma)}{\partial\xi_b}\right]\right)\right), \tag{86}
$$

where [ ] denotes the extent of applicability of the functional derivative within, $G_{\Gamma\Delta}$ is an invertible ultralocal kinetic metric and Π^{Γ} is the momentum conjugate to Ψ_Δ.
ii) If, additionally, the potential contains covariant derivatives, then there is an extra contribution to i):

$$
\frac{2\sqrt{g}}{N}\nabla_b\left(N^2p_{ij}\left(\frac{\partial^{\Psi}U}{\partial\Gamma^a{}_{ic}}g^{aj} - \frac{1}{2}\frac{\partial^{\Psi}U}{\partial\Gamma^a{}_{ij}}g^{ac}\right)\right). \tag{87}
$$

The proofs offered here include both conformal gravity ($s = 1$) and strong conformal gravity ($s = 0$, $\Lambda \neq 0$). There is an equivalent derivation which encompasses GR and the arbitrary-W strong gravity theories. Result i) in the GR case is related to a result of Teitelboim [41] that the contributions of nonderivatively-coupled fields to the Hamiltonian and momentum constraints independently satisfy the Dirac algebra. In the working below, this is reflected by our ability to split the working into pure gravity and matter parts.

Use of formulae i), ii) permits the ${}^\Psi\dot{\mathcal{H}}^{\mathrm{C}}$ calculations to be done without explicitly computing the Euler–Lagrange equations. This is because our derivation uses once and for all the *general* Euler–Lagrange equations.

Proof

i) For a homogeneous quadratic kinetic term ${}^\Psi T = (\dot{\Psi}_\Gamma - \pounds_\xi \Psi_\Gamma)(\dot{\Psi}_\Delta - \pounds_\xi \Psi_\Delta) G^{\Gamma\Delta}\left(\frac{V^{\frac{2}{3}}}{\phi^4} g^{ij}\right)$, the conjugate momenta are $\Pi^\Delta = \frac{\partial L}{\partial \dot{\Psi}_\Delta} = \frac{\sqrt{g}\phi^4}{2NV^{\frac{2}{3}}} G^{\Gamma\Delta}(\dot{\Psi}_\Gamma - \pounds_\xi \Psi_\Gamma)$.

The ξ^i-variation gives the momentum constraint

$$-{}^\Psi\mathcal{H}_i^{\mathrm{C}} \equiv 2\nabla_j p_i{}^j - \Pi^\Delta \frac{\delta(\pounds_\xi \Psi_\Delta)}{\delta \xi^i} = 0 \tag{88}$$

and the local square root gives the Hamiltonian constraint, which is

$$-{}^\Psi\mathcal{H}^{\mathrm{C}} \equiv \frac{\sqrt{g}}{V^{\frac{2}{3}}}(sR + {}^\Psi U) - \frac{V^{\frac{2}{3}}}{\sqrt{g}}(p^{ij}p_{ij} + G_{\Delta\Gamma}\Pi^\Gamma\Pi^\Delta) = 0 \tag{89}$$

in the distinguished representation. Then

$$-{}^\Psi\dot{\mathcal{H}}^{\mathrm{C}} \approx \tfrac{\sqrt{g}}{V^{\frac{2}{3}}}(s\dot{R} + {}^\Psi\dot{U}) - \tfrac{2V^{\frac{2}{3}}}{\sqrt{g}}(\dot{p}^{ij}p_{ij} + \dot{g}_{ik}p^{ij}p^k{}_j) - \tfrac{V^{\frac{2}{3}}}{\sqrt{g}}(2\dot{\Pi}^\Delta G_{\Gamma\Delta}\Pi^\Gamma + \dot{G}_{\Gamma\Delta}\Pi^\Delta\Pi^\Gamma), \tag{90}$$

using the chain-rule on (89) and using $\dot{g} = \dot{V} = 0$. Now use the chain-rule on ${}^\Psi\dot{U}$, the Euler–Lagrange equations $\dot{p}^{ij} = \frac{\delta L}{\delta g_{ij}}$ and $\dot{\Pi}^\Delta = \frac{\delta L}{\delta \Psi_\Delta}$, and $p = 0$ to obtain the first step below:

$$\begin{aligned}
-{}^\Psi\dot{\mathcal{H}}^{\mathrm{C}} \approx\; & \tfrac{\sqrt{g}s}{V^{\frac{2}{3}}}\dot{R} + \tfrac{\sqrt{g}}{V^{\frac{2}{3}}}\left(\tfrac{\partial^\Psi U}{\partial \Psi_\Delta}\dot{\Psi}_\Delta + \tfrac{\partial^\Psi U}{\partial(\nabla_b\Psi_\Delta)}(\nabla_b\dot{\Psi}_\Delta) + \tfrac{\partial^\Psi U}{\partial g_{ab}}\dot{g}_{ab}\right)\\
& -2sp^{ij}\left[\tfrac{\delta R}{\delta g_{ij}}N\right] - 2p^{ij}\left[\tfrac{\delta^\Psi U}{\delta g_{ij}}N\right] - \tfrac{1}{2N}p^{ij}\tfrac{\partial^\Psi T}{\partial g_{ij}} - \tfrac{4NV^{\frac{2}{3}}}{g}p_{ik}p^{ij}p^k{}_j\\
& -2G_{\Gamma\Delta}\Pi^\Gamma\left[N\tfrac{\delta U_\Psi}{\delta\psi_\Delta}\right] - \tfrac{V^{\frac{2}{3}}}{\sqrt{g}}\dot{G}_{\Delta\Gamma}\Pi^\Delta\Pi^\Gamma\\
=\; & \left(\tfrac{\sqrt{g}s}{V^{\frac{2}{3}}}\dot{R} - 2sp^{ij}\left[\tfrac{\delta R}{\delta g_{ij}}N\right] - \tfrac{4NV^{\frac{2}{3}}}{g}p_{ik}p^{ij}p^k{}_j\right)\\
& + \tfrac{\sqrt{g}}{V^{\frac{2}{3}}}\left(\tfrac{\partial^\Psi U}{\partial\Psi_\Delta}\left(\tfrac{2NV^{\frac{2}{3}}}{\sqrt{g}}\Pi^\Gamma G_{\Gamma\Delta}\right) + \tfrac{\partial^\Psi U}{\partial(\nabla_b\Psi_\Delta)}\nabla_b\left(\tfrac{2NV^{\frac{2}{3}}}{\sqrt{g}}\Pi^\Gamma G_{\Gamma\Delta}\right) + \tfrac{\partial^\Psi U}{\partial g_{ab}}\left(\tfrac{2NV^{\frac{2}{3}}}{\sqrt{g}}p_{ab}\right)\right)\\
& -2p^{ij}\tfrac{\partial^\Psi U}{\partial g_{ij}}N - \tfrac{1}{2N}p^{ab}\tfrac{\partial G^{\Delta\Gamma}}{\partial g_{ab}}\dot{\Psi}_\Gamma\dot{\Psi}_\Delta\\
& -2G_{\Gamma\Delta}\Pi^\Gamma N\tfrac{\partial^\Psi U}{\partial\psi_\Delta} + 2G_{\Gamma\Delta}\Pi^\Gamma\nabla_b\left(N\tfrac{\partial^\Psi U}{\partial(\nabla_b\Psi_\Delta)}\right) - \tfrac{V^{\frac{2}{3}}}{\sqrt{g}}\tfrac{\partial G_{\Delta\Gamma}}{\partial g_{ij}}\dot{g}_{ij}\Pi^\Delta\Pi^\Gamma\\
\approx\; & \tfrac{s}{N}\nabla_b\left(N^2\Pi^\Delta\tfrac{\delta(\pounds_\xi\Psi_\Delta)}{\delta\xi_b}\right) + \sqrt{g}\tfrac{\partial^\Psi U}{\partial(\nabla_b\Psi_\Delta)}\nabla_b\left(\tfrac{2N}{\sqrt{g}}\Pi^\Gamma G_{\Delta\Gamma}\right) + 2G_{\Gamma\Delta}\Pi^\Gamma\nabla_b\left(N\tfrac{\partial^\Psi U}{\partial(\nabla_b\Psi_\Delta)}\right).
\end{aligned} \tag{91}$$

In the second step above, we regroup the terms into pure gravity terms and matter terms, expand the matter variational derivatives and use the definitions of the momenta to eliminate the velocities in the first three matter terms. We now observe that the first and sixth matter terms cancel, as do the third and fourth. In the third step we use the pure gravity working and the momentum constraint (88), and the definitions of the momenta to cancel the fifth and eight terms of step 2. Factorization of step 3 gives the result.

ii) Now $-{}^{\Psi}\dot{\mathcal{H}}^{\mathrm{C}}$ has 2 additional contributions in step 2 due to the presence of the connections:

$$\frac{\sqrt{g}}{V^{\frac{2}{3}}}\frac{\partial U_\Psi}{\partial \Gamma^a{}_{bc}}\dot{\Gamma}^a{}_{bc} - 2p^{ij}\left[\frac{\partial U_\Psi}{\partial \Gamma^a{}_{bc}}\frac{\delta \Gamma^a{}_{bc}}{\delta g_{ij}}N\right], \tag{92}$$

which, using

$$\delta\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(\nabla_c(\delta g_{db}) + \nabla_b(\delta g_{dc}) - \nabla_d(\delta g_{bc})), \tag{93}$$

$$\dot{\Gamma}^a{}_{bc} = \frac{V^{\frac{2}{3}}}{2\sqrt{g}}(\nabla_b(Np_c{}^a) + \nabla_c(Np_b{}^a) - \nabla^a(Np_{bc})), \tag{94}$$

integration by parts on the second term of (92) and factorization yields ii). □

Although Theorem 1 does not consider potentials containing Christoffel symbols, in all the cases that we consider below (which suffice for the investigation of the classical bosonic theories of nature) the propagation of ${}^{\Psi}\mathcal{H}^{\mathrm{C}}$ rules out all theories with such potentials. Thus it is not an issue whether such theories permit $p = 0$ to be propagated.

7 Conformal Gravity Coupled to Matter Fields: Examples

In this section we take $s = 1$ for Lorentzian (as opposed to Euclidean or strong) conformal gravity. We will also use $W = 0$ from the outset, and $\Lambda = 0$, so that we are investigating whether our theory of pure conformal gravity is capable of accommodating conventional classical matter theories and establishing the physical consequences. We find that it does, and that the known classical bosonic theories are singled out.

7.1 Scalar Field

The natural action to consider according to our prescription for including a scalar field is

$${}^{\psi}S = \int \mathrm{d}\lambda \int \mathrm{d}^3x \left(\left(\frac{\sqrt{g}\phi^6}{V}\right)\sqrt{\left(\frac{V^{\frac{2}{3}}}{\phi^4}\right)\left(R - \frac{8\nabla^2\phi}{\phi} + {}^{\psi}U_{(1)}\right) + {}^{\psi}U_{(0)}}\sqrt{T^{\mathrm{C}} + {}^{\psi}T}\right) \tag{95}$$

$$= \int d\lambda \frac{\int \mathrm{d}^3x\sqrt{g}\phi^4\sqrt{R - \frac{8\nabla^2\phi}{\phi} + {}^{\psi}U_{(1)} + \frac{{}^{\psi}U_{(0)}\phi^4}{V(\phi)^{\frac{2}{3}}}}\sqrt{T^{\mathrm{C}} + {}^{\psi}T}}{V(\phi)^{\frac{2}{3}}} = \int \mathrm{d}\lambda \frac{\bar{I}}{V^{\frac{2}{3}}}, \tag{96}$$

where, once again, we give two different expressions to exhibit the homogeneity and to use in calculations. ${}^{\psi}U_{(0)}$ is an arbitrary function of ψ alone whilst ${}^{\psi}U_{(1)} = -\frac{C}{4}g^{ab}(\partial_a\psi)\partial_b\psi$. $\sqrt{C}$ is the a priori unfixed canonical speed of propagation of the scalar field.

The conjugate momenta p^{ij} and p_ϕ are given by (35) and (34) but with

$$2N = \sqrt{\frac{T^{\mathrm{C}} + {}^{\psi}T}{R - \frac{8\nabla^2\phi}{\phi} + {}^{\psi}U_{(1)} + \frac{{}^{\psi}U_{(0)}\phi^4}{V(\phi)^{\frac{2}{3}}}}},$$

and additionally we have the momentum conjugate to ψ, $p_\psi = \frac{\sqrt{g}\phi^4}{2N}(\dot{\psi} - \pounds_\xi\psi)$. As in the case of pure conformal gravity, we have the primary constraint (36), and the end-point part of the ϕ-variation gives $p_\phi = 0$, so that $p = 0$ by the primary constraint. But by construction (Theorem 1) this action has the correct form to propagate the constraint $p = 0$ provided the lapse-fixing equation

$$2(NR - \nabla^2 N) + \frac{3N^{\psi}U_{(0)}}{V^{\frac{2}{3}}} + 2N^{\psi}U_{(1)} = \frac{1}{V^{\frac{5}{3}}}\int \mathrm{d}^3x\sqrt{g}N^{\psi}U_{(0)} + \frac{\bar{I}}{V}, \tag{97}$$

holds (in the distinguished representation), but this is guaranteed from the rest of the ϕ-variation. The ξ^i-variation gives the momentum constraint $-{}^{\psi}\mathcal{H}_i^{\mathrm{C}} \equiv 2\nabla_j p_i{}^j - p_{\psi}\partial_i\psi = 0$,
whilst the local square root gives rise to the Hamiltonian constraint, which is

$$-{}^{\psi}\mathcal{H}^{\mathrm{C}} \equiv \frac{\sqrt{g}}{V^{\frac{2}{3}}}\left(R + {}^{\psi}U_{(1)} + \frac{{}^{\psi}U_{(0)}}{V^{\frac{2}{3}}}\right) - \frac{V^{\frac{2}{3}}}{\sqrt{g}}(p^{ij}p_{ij} + p_{\psi}^2) = 0 \tag{98}$$

in the distinguished representation.

Then, using formula i), the propagation of the Hamiltonian constraint is, weakly,

$${}^{\psi}\dot{\mathcal{H}}^{\mathrm{C}} \approx \frac{(C-1)}{N}\nabla_b(N^2 p_{\psi}\partial^b\psi). \tag{99}$$

Now, if the cofactor of $(C-1)$ were zero, there would be a secondary constraint which would render the scalar field theory trivial by using up its degree of freedom. Hence $C = 1$ is fixed, which is the universal light-cone condition applied to the scalar field. This means that the same light cone as that of the gravitation is enforced even though the gravitational theory in question is not generally covariant.

7.2 1-Form Fields

According to our prescription, the natural action to include electromagnetism is

$${}^{\mathrm{A}}S = \int \mathrm{d}\lambda \int \mathrm{d}^3x\left(\left(\frac{\sqrt{g}\phi^6}{V}\right)\sqrt{\left(\frac{V^{\frac{2}{3}}}{\phi^4}\right)\left(R - \frac{8\nabla^2\phi}{\phi}\right) + \left(\frac{V^{\frac{4}{3}}}{\phi^8}\right){}^{\mathrm{A}}U}\sqrt{T^{\mathrm{C}} + \left(\frac{V^{\frac{2}{3}}}{\phi^4}\right){}^{\mathrm{A}}T}\right) \tag{100}$$

$$= \int \mathrm{d}\lambda \frac{\int \mathrm{d}^3x\sqrt{g}\phi^4\sqrt{R - \frac{8\nabla^2\phi}{\phi} + {}^{\mathrm{A}}U\frac{V(\phi)^{\frac{2}{3}}}{\phi^4}}\sqrt{T^{\mathrm{C}} + {}^{\mathrm{A}}T\frac{V(\phi)^{\frac{2}{3}}}{\phi^4}}}{V(\phi)^{\frac{2}{3}}} = \int \mathrm{d}\lambda\frac{\bar{I}}{V^{\frac{2}{3}}}, \tag{101}$$

for

$${}^{\mathrm{A}}U = -\hat{C}(\nabla_b A_a - \nabla_a A_b)\nabla^b A^a, \tag{102}$$

$${}^{\mathrm{A}}T = g^{ab}(\dot{A}_a - \pounds_{\xi}A_a)(\dot{A}_b - \pounds_{\xi}A_b). \tag{103}$$

We will first show that electromagnetism exists as a theory coupled to conformal gravity. We will then discuss how it is uniquely picked out (much as it is picked out in RWR [11]), and how Yang–Mills theory is uniquely picked out upon consideration of K interacting 1-form fields (much as it is picked out in [12]).

Again, the conjugate momenta p_{ϕ} and p^{ij} are given by (34) and (35) but now with

$$2N = \sqrt{\frac{T^{\mathrm{C}} + \frac{{}^{\mathrm{A}}TV(\phi)^{\frac{2}{3}}}{\phi^4}}{s\left(R - \frac{8\nabla^2\phi}{\phi}\right) + \frac{{}^{\mathrm{A}}UV(\phi)^{\frac{2}{3}}}{\phi^4}}},$$

and additionally we have the momentum conjugate to A_i, $\pi^i = \frac{\sqrt{g}}{2N}g^{ij}(\dot{A}_j - \pounds_{\xi}A_j)$.
By the same argument as in previous sections, $p = 0$ arises and is preserved by a lapse-fixing equation, which is now

$$2(NR-\nabla^2N)+\left(N\,{}^{\mathrm{A}}U-\frac{{}^{\mathrm{A}}T}{4N}\right)V^{\frac{2}{3}}+\frac{1}{V^{\frac{1}{3}}}\int \mathrm{d}^3x\sqrt{g}\left(N\,{}^{\mathrm{A}}U+\left(\frac{{}^{\mathrm{A}}T}{4N}\right)\right)=\frac{\bar{I}}{V} \qquad (104)$$

The ξ^i-variation gives the momentum constraint $-{}^{\mathrm{A}}\mathcal{H}_i^{\mathrm{C}}\equiv 2\nabla_j p_i{}^j-(\pi^c(\nabla_iA_c-\nabla_cA_i)-\nabla_c\pi^cA_i)=0$, whilst the local square root gives rise to the Hamiltonian constraint, which is

$$-{}^{\mathrm{A}}\mathcal{H}^{\mathrm{C}}\equiv\frac{\sqrt{g}}{V^{\frac{2}{3}}}\left(sR+{}^{\mathrm{A}}UV^{\frac{2}{3}}\right)-\frac{V^{\frac{2}{3}}}{\sqrt{g}}(p^{ij}p_{ij}+\frac{1}{V^{\frac{2}{3}}}\pi_i\pi^i)=0 \qquad (105)$$

in the distinguished representation.

Then, using formula i), the propagation of the Hamiltonian constraint is, weakly,

$$-{}^{\mathrm{A}}\dot{\mathcal{H}}^{\mathrm{C}}\approx\frac{1}{N}\nabla^b\left(N^2\left((1-4\hat{C})\pi^i(\nabla_bA_i-\nabla_iA_b)-A_b\nabla_i\pi^i\right)\right) \qquad (106)$$

Suppose the cofactor of $1-4\hat{C}$ is zero. Then we require $\nabla_{[b}A_{i]}=0$. But this is three conditions on A_i, so the vector theory would be rendered trivial. Thus, exhaustively, the only way to obtain a consistent theory is to have the universal light-cone condition $\hat{C}=1/4$ and the new constraint

$$\mathcal{G}\equiv\nabla_a\pi^a=0, \qquad (107)$$

which is the electromagnetic Gauss constraint. The propagation of $\mathcal{G}$ is no further bother because the A_i Euler–Lagrange equation

$$\frac{\mathrm{d}\pi^i}{\mathrm{d}\lambda}=2\sqrt{g}C\nabla_b(\nabla^bA^i-\nabla^iA^b) \qquad (108)$$

is free of V and hence identical to that in the RWR case. Since the RWR argument for the propagation of $\mathcal{G}$ follows from (108), this guarantees that the result also holds in conformal gravity.

We can finally encode this new constraint by making use of the best matching associated with the U(1) symmetry of the potential, to modify the kinetic term (103) by the introduction of an auxiliary variable Φ to

$${}^{\mathrm{A}}T=(\dot{A}_a-\pounds_\xi A_a-\partial_a\Phi)(\dot{A}^a-\pounds_\xi A^a-\partial^a\Phi). \qquad (109)$$

The following extensions of this working have been considered.

1) Additionally, replacing ${}^{\mathrm{A}}\,U$ by $C^{abcd}\nabla_bA_a\nabla_dA_c$ for $C^{abcd}=C_1g^{ac}g^{bd}+C_2g^{ad}g^{bc}+C_3g^{ab}g^{cd}$ in the action preserves the correct form to guarantee $p=0$ is maintained. We now have derivative coupling contributions also, so we need to make use of formula ii) of theorem 2 as well as formula i). Thus, weakly

$$-{}^{\mathrm{A}}\dot{\mathcal{H}}^{\mathrm{C}}\approx\frac{1}{N}\nabla_b\left(N^2\left(\begin{array}{l}(4C_1+1)\pi^a\nabla^bA_a+(4C_2-1)\pi^a\nabla_aA^b+4C_3(N^2\pi^b\nabla_aA^a)\\ -\nabla_a\pi^aA^b-4p_{ij}\nabla_{(d}A_{b)}\left(C^{ajbd}A^i-\frac{1}{2}C^{ijbd}A^a\right).\end{array}\right)\right) \qquad (110)$$

This has the same structure in A_i as for the GR case [the overall $V^{-\frac{2}{3}}$ is unimportant, as is the replacement of the GR $(p_{ij}-\frac{p}{2}g_{ij})$ factors by (p_{ij}) factors here], so an argument along the same lines as that used in RWR will hold, forcing the Gauss constraint and $C_1=-C_2=-\frac{1}{4}$, $C_3=0$ (Maxwell theory).

2) The changes

$${}^{\mathrm{A}}T\longrightarrow\ {}^{\mathrm{A}_I}T=g^{ij}(\dot{A}_i^I-\pounds_\xi A_i^I)(\dot{A}_{jI}-\pounds_\xi A_{jI}), \qquad (111)$$

$${}^{\mathrm{A}}U\longrightarrow{}^{\mathrm{A}_I}U=O_{IK}C^{abcd}\nabla_bA_a^I\nabla_dA_c^K+B^I{}_{JK}\bar{C}^{abcd}\nabla_bA_{Ia}A_c^JA_d^K+I_{JKLM}\bar{\bar{C}}^{abcd}A_a^JA_b^KA_c^LA_d^M \qquad (112)$$

(for a priori distinct supermetrics C, $\bar{C}$, $\bar{\bar{C}}$) to the ansatz preserve the conformal properties, hence guaranteeing that $p = 0$ is maintained by the lapse-fixing equation obtained by applying (111, 112) to (104). The new conjugate momenta are $\pi^i_I = \frac{\sqrt{g}}{2N} g^{ij}(\dot{A}_{Ij} - \pounds_\xi A_{Ij})$.
The ξ^i-variation gives the momentum constraint,

$$-^{\mathrm{A}_I}\mathcal{H}_i^{\mathrm{C}} = 2\nabla_j p^j{}_i - (\pi^{Ij}(\nabla_i A_{Ij} - \nabla_j A_{Ii}) - \nabla_j \pi^j_I A^I_i) = 0, \tag{113}$$

and the local square root gives the Hamiltonian constraint, which is

$$-^{\mathrm{A}_I}\mathcal{H}^{\mathrm{C}} \equiv \frac{\sqrt{g}}{V^{\frac{2}{3}}}\left(sR + U_{\mathrm{A}_I} V^{\frac{2}{3}}\right) - \frac{V^{\frac{2}{3}}}{\sqrt{g}}(p^{ij}p_{ij} + \frac{1}{V^{\frac{2}{3}}}\pi^I_i \pi^i_I) = 0 \tag{114}$$

in the distinguished representation.

Using formulae i), ii) we read off that the propagation of the Hamiltonian constraint is, weakly,

$$-^{\mathrm{A}_I}\dot{\mathcal{H}}^{\mathrm{C}} \approx \frac{1}{N}\nabla_b\left(N^2 \left(\begin{array}{l} (4C_1 O^{IK} + \delta^{IK})\pi^a_I \nabla^b A_{Ka} + (4C_2 O^{IK} - \delta^{IK})\pi^a_I \nabla_a A^b_K \\ +4C_3 O^{IK}\pi^b_I \nabla_a A^a_K - \left(\nabla_a \pi^a_K A^{Kb} - 2\bar{C}^{abcd} B^I{}_{JK}\pi_{Ia} A^J_c A^K_d\right) \\ -2O^{IK} p_{ij} \nabla_{(d|} A_{K|l)}(2A^i_I C^{bjld} - A^b_I C^{ijld}) \\ -B^I{}_{JK} p_{ij} A^J_l A^K_d (2A^i_I \bar{C}^{bjld} - A^b_I \bar{C}^{ijld}) \end{array} \right)\right). \tag{115}$$

In the same sense as for the single vector field case above, (115) has the same structure as for the GR case, so the argument used in [12] will hold, forcing

$$O^{IK} = \delta^{IK}, C_1 = -C_2 = -\frac{1}{4}, C_3 = 0, \bar{C}_3 = 0, \tag{116}$$

$$B_{I(JK)} = 0 \Leftrightarrow \bar{C}_1 = -\bar{C}_2 \equiv -\frac{\mathtt{g}}{4} \tag{117}$$

(for some emergent coupling constant $\mathtt{g}$) and leaving the new constraint

$$\mathcal{G}_J \equiv \nabla_a \pi^a_J - \ \mathtt{g} B_{IJK}\pi^I_a A^{Ka}. \tag{118}$$

Again as for the single vector field case, the π^a_J Euler–Lagrange equation is unchanged from the GR case. The action of the dot on A_{Ka} gives no volume terms. Hence the working for the propagation of $\mathcal{G}_J$ is unchanged from that in [12], which enforces the following conditions:

$$I_{JLKM} = B^I{}_{JK} B_{ILM}, \bar{\bar{C}}_2 = -\bar{\bar{C}}_1 = \frac{\mathtt{g}^2}{16}, \bar{\bar{C}}_3 = 0, \tag{119}$$

$$B^I{}_{JK} B_{ILM} + B^I{}_{JM} B_{IKL} + B^I{}_{JL} B_{IMK} = 0 \text{ (Jacobi identity)}, \tag{120}$$

$$B_{IJK} = B_{[IJK]} \text{ (total antisymmetry)}. \tag{121}$$

From (117) and (120), it follows that the B_{IJK} are the structure constants of some Lie algebra, $\mathcal{A}$. From (121) and the Gell-Mann–Glashow theorem [42], $\mathcal{A}$ is the direct sum of compact simple and $U(1)$ subalgebras, provided that the kinetic term is positive definite as assumed here. We can defend this assumption because we are working on a theory in which even the gravitational kinetic term is taken to be positive definite; positive-definite kinetic terms ease quantization.

3) Mass terms are banned by the propagation of the Gauss laws. Mass terms contain nontrivial powers of the volume; however the above arguments can easily be extended to accommodate them. In the many vector fields case, the effect of a mass term is to give rise to a new term $\frac{2N}{V^{\frac{2}{3}}} M^{JK} A^i_K$ in the Euler–Lagrange equations, which contributes a term $2M^{JK}\nabla_i\left(\frac{N}{V^{\frac{2}{3}}} A^i_K\right)$ to the propagation of $\mathcal{G}_J$. For this to vanish, either $A^i_K = 0$ which renders the vector theory trivial, or $M^{JK} = 0$.

8 Discussion

The differences between the conformal-gravity–matter and GR–matter metric Euler–Lagrange equations are the absence of a term containing the 'expansion of the universe' p and the presence of a global term such as

$$-g^{ab}\frac{\sqrt{g}}{3V^{\frac{2}{3}}}\left\langle N\left(2(R+{}^{\psi}U_{(1)})+3\frac{{}^{\psi}U_{(0)}}{V^{\frac{2}{3}}}\dots\right)\right\rangle \tag{122}$$

for scalar matter. Such a global term mimics the effect of a small epoch-dependent cosmological constant. This global term is a 'cosmological force' because it occurs in the Euler–Lagrange equations with proportionality to g^{ab}, just like the cosmological constant contribution does in GR. We expect it to be epoch-dependent because it contains matter field contributions, which will change as the universe evolves. The occurrence of this global term should be compared with the particle model in PD, in which there is a universal cosmological force induced by all the familiar forces of nature such as Newtonian gravity and electrostatics. We have seen that in the particle model this cosmological force is extremely weak over solar system scales but has a decisive effect on cosmological scales, ensuring the conservation of the moment of inertia, I.

Similarly, we expect that conformal gravity will reproduce the the solar-system and the binary-pulsar results[14] just as well as GR. This is because, first, the expansion of the universe does not play a role on such small scales in GR so its absence will not affect the results. Second, at maximal expansion, a data set may be evolved by both the GR and conformal gravity equations. The difference between these two evolutions is well defined in Riem. Since the first derivatives match up at maximal expansion, the difference between the evolutions is small. For sure, the size of the difference will depend on the global terms. But these can be made small by a well-known construction, as far as the finite-time evolution for a patch of initial data that is substantially smaller than the radius of the universe is concerned. Such patches can be constructed to contain the solar system or the region containing both the solar system and the binary pulsar. Clearly these arguments will not apply to cosmology, for which the differences between GR and conformal gravity must have dramatic consequences.

At this stage of our work, we have only just started to explore these consequences. *Prima facie*, it does seem unlikely that conformal gravity will be able to supplant the Big Bang cosmology, on account of the strong evidence from the Hubble redshift, nucleosynthesis and the microwave background. Prior to further detailed study, we refer the reader to the comments made at the end of PD. Instead, we should like to consider the potential value of conformal gravity as a foil to the Big Bang. Theorists concerned with achieving the deepest possible understanding of cosmology and the foundations of physics value alternative models [43], even if they explain or mimic only part of the whole picture. For decades the Brans–Dicke alternative has played an invaluable role, and, in its present guise as a dilaton field, it is currently actually more orthodox than GR. Seen in this light, conformal gravity and the reinterpretation of the CMC-sliceable solutions of GR as geodesics on CS+V have several positive features.

Above all, they represent a new and radical approach to scale invariance. They show that best matching and constraint propagation are powerful tools in theory construction. In particular, they highlight the thought-provoking manner in which GR only just fails to be fully scale invariant.

Another potential strength of conformal gravity is that it forces one to consider cosmology in a more sophisticated manner. Consider the isotropic and homogeneous FRW cosmologies, which are the backbone of the standard model. As self-similar solutions in which nothing changes except size and homogeneous intensity, they must raise doubts. From the dynamical point of view, they

[14] We require further study of the weak field limit to be quantitatively sure about the binary pulsar.

are suspiciously trivial. In a scale-invariant theory, the FRW solutions are merely static points in the configuration space. There has long been concern [44] about the accuracy with which they approximate more physically realistic inhomogeneous solutions of GR under the assumption that it is the correct physical theory. Conformal gravity raises a more serious doubt – GR might be spectacularly wrong for cosmology despite being wonderfully accurate for all other applications. In comparison, the dilatonic modifications of GR [29] have no significant effect on the key physical basis of the big-bang scenario– the explanation of the Hubble red shift by actual expansion of the universe.

Since conformal gravity has no dynamics analogous to the FRW universes of GR, the only possible progress in its cosmology will be through the study of inhomogeneous solutions. This is the opposite emphasis to the norm in classical and quantum cosmology and does have some chance to throw up a radical new explanation of the red shift. We know that a change in clumpiness (shape) of the universe can cause redshift in GR. The solar photons that reach us are redshifted by having to climb out of the solar gravitational potential well (gravitational redshift), and inhomogeneities cause similar effects in cosmology (the integrated Sachs–Wolfe and Rees–Sciama effects [46]). The particle model in PD is suggestive in this respect. We speculate that the rearrangement of geometry and matter of an evolving universe can cause a similar redshift in conformal gravity. In such a case, it will not be due to differences in the gravitational potential between different points of space but between different epochs. Now, as pointed out in PD, the potential can be changed either by a change of scale or by a change of shape. Conformal gravity suggests the former is not available and that the latter is the origin of the Hubble redshift. Since the change of shape of the universe can be observed, this should lead to testable predictions.

Another service that conformal gravity can perform is to stimulate a thorough reexamination of the problem of singularities. The Big Bang itself is an initial singularity where the known laws of physics break down. It is inevitable in GR by theorems of Hawking [45]. These require the expansion of past-directed normal timelike geodesic congruences to be positive everywhere on a given spatial hypersurface. The GR form of these theorems will not hold in conformal gravity since such a notion of expansion is no longer meaningful.[15] In GR, the Hubble redshift interpretation forces one to admit the breakdown of known physics in our finite past, whilst in conformal gravity, the denial of such a breakdown must be accompanied by a new interpretation of the Hubble redshift.

Whereas our greatest interest is in whether conformal gravity can give us an alternative cosmology, our CS+V theory has a notion of universal expansion, so it will be much closer to GR both in agreeing with the standard cosmology and in not offering these new perspectives on nonsingularity and global cosmological forces.

We finish by discussing quantization. For conformal gravity, 1) we hope to quantize in the timeless interpretation due to one of the authors [47]. 2) The Hamiltonian constraint adopts a new role in conformal gravity since it no longer uses up a degree of freedom. 3) This and the fundamental lapse-fixing equation (39) are nonstandard objects from the quantization perspective. 4) The new global terms may play a role. 5) Finally, whereas in GR the DeWitt supermetric gives an indefinite inner product as a consequence of the sign of the expansion contribution to the kinetic energy, in conformal gravity the new $W = 0$ supermetric gives a positive definite inner product. This ameliorates the inner product problem [48] of quantum gravity. Notice that 1) – 3) will still be features of the quantum CS+V theory.

[15] We do not know if other forms of singularity theorem hold. We cannot so easily dismiss results involving null and/or local expansion. Another source of trouble in adapting GR proofs for conformal gravity will be the lack of an equivalent to the Efe's. If local singularities form, they will contribute to the global terms in conformal gravity. This could be fatal in the particle model, but in conformal gravity there may be two ways out. First, singularities may only contribute a finite amount once integrated. Second, in GR, there is the 'collapse of the lapse' in approaching local singularities, which is a gauge effect; the analogue of this in conformal gravity would be a real physical effect.

We have a nice set of stepping-stones toward quantization of conformal gravity. The effect of the $W = 0$ supermetric on quantization can be tried out by itself in the strong gravity regime, since one of us found this to be a consistent theory [16]. Then the additional effect of introducing a volume and having $\mathcal{H}^{\rm C}$ no longer use up a degree of freedom can be tried for strong conformal gravity, since for this the additional complication of the lapse-fixing equation interpretation of full conformal gravity is trivially absent (since N is a spatial constant).

That conformal gravity has a marginally smaller configuration space than GR makes our quantum program attractive. We hope to use a 'top-down' approach: to start from firm classical theory and deduce features of the quantum universe. However, we start from space rather than spacetime for relational reasons [13, 47, 11, 9] and to illustrate the potential naivety of presupposing and extending spacetime structure. The great problems of quantizing gravity are hopelessly interrelated, so that adding to a partial resolution to tackle further problems can spoil that partial resolution [48]. So it is not to be expected that Ashtekar variable techniques [49], with their resolution of operator ordering and their natural regularization, could be imported into conformal gravity. Thus, quantization of conformal gravity will differ from, but not necessarily be easier than, quantization of GR. Should conformal gravity adequately describe the classical universe, its quantization program will become of utmost importance. Even if this were not the case, we expect to further the understanding of quantization and of quantum general relativity by such a program.

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